

# Spatial Discretization of Axially Moving Media Vibration Problems

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*Spatial discretization of axially moving media eigenvalue problems is examined from the perspectives of moving versus stationary system basis functions, configuration space versus state space form discretization, and subcritical versus supercritical speed convergence. The moving string eigenfunctions, which have previously been shown to give excellent discretization convergence under certain conditions, become linearly dependent and cause numerical problems as the number of terms increases. This problem does not occur in a discretization of the state space form of the eigenvalue problem, although convergence is slower, not monotonic, and not necessarily from above. Use of the moving string basis at supercritical speeds yields strikingly poor results with either the configuration or state space discretizations. The stationary system eigenfunctions provide reliable eigenvalue predictions across the range of problems examined. Because they have known exact solutions, the moving string on elastic foundation and the traveling, tensioned beam are used as illustrative examples. Many of the findings, however, apply to more complex moving media problems, including nontrivial equilibria of nonlinear models. [S0739-3717(00)02103-6]*

## Introduction

Magnetic tape drives, power transmission belts, pipes conveying fluids and fibers in textile manufacturing are some examples of axially moving media. Analytical solutions for the vibration of these systems are difficult to find in general, and discretization methods are frequently used to obtain the natural frequency spectra and vibration modes. In the literature, the most common approach has been to discretize the configuration space form equation of motion (that is, the governing equation for the displacement field) using the corresponding stationary system eigenfunctions [1–4]. Parker's [4] results demonstrate the radically incorrect eigenvalue predictions that this discretization basis can give at supercritical speeds. Wickert and Mote [5] showed that use of the complex, speed-dependent eigenfunctions of a related gyroscopic system as basis functions can significantly improve eigenvalue predictions. Chen [6] adopted this idea and used the moving string eigenfunctions to discretize an axially moving string coupled to a mass-spring-damper system. While both approaches use the complex, moving string eigenfunction basis, they yield quite different results. Wickert and Mote's [5] approach is equivalent to discretization of the configuration space eigenvalue problem, and Chen's [6] discretization is applied to a state space formulation of the eigenvalue problem. A critical comparison of these approaches against known, exact solutions was not attempted by these or other researchers for sub- or supercritical speeds. Furthermore, use of complex, speed-dependent basis functions requires special care, a thorough discussion of which is lacking in the literature.

This study presents a systematic analysis of the spatial discretization of axially moving media problems. The problem is examined from the perspectives of configuration and state space form discretizations, stationary versus moving system basis functions, and subcritical versus supercritical speed convergence. Configuration and state space discretizations are shown to yield markedly different results, and the limitations of each are discussed. The moving string eigenfunctions, which are generally thought to be the superior basis for moving string problems, can experience potentially severe numerical problems due to their linear dependence. Furthermore, this basis is apparently incomplete at super-

critical speeds. In fact, no acceptable approach was found for supercritical speed moving string problems, and published discretization results for such cases should be viewed with caution. The findings are illustrated by two examples having known, exact solutions: 1) moving string on an elastic foundation, and 2) moving beam on simple supports.

The intent is to assess discretization methods rather than physical behavior of moving media systems, so the examples are limited to systems for which exact solutions are available. The findings, however, apply to a broad range of unsolved problems for which discretization might be used. For instance, nonlinear effects that are crucial at high-speed lead to bifurcated, nontrivial equilibria; the results herein identify key issues for eigenvalue problem discretizations about such equilibria.

## Configuration and State Space Discretizations

The linearized, nondimensional equation of motion for free vibration of an axially moving medium is

$$Mw_{tt} + Gw_t + Kw = 0 \quad (1)$$

where  $M$ ,  $G$  and  $K$  are linear, time-invariant operators and  $w(P, t)$  is the displacement field. With the inner product  $(u, v) = \int_P u \bar{v} dP$ ,  $M$  and  $K$  are self-adjoint while  $G$  is skew-self-adjoint (the overbar denotes complex conjugate). The eigenvalue problem obtained from  $w(P, t) = u(P)e^{i\omega t}$  is

$$(-\omega^2 M + i\omega G + K)u = 0 \quad (2)$$

This is referred to as the *configuration space form* in this work.

Let  $\beta_j, \varphi_j(P)$  be the known  $j$ th eigensolution pair of a related, simpler gyroscopic system. The  $\beta_j$  are the real natural frequencies for subcritical speeds, and the  $\varphi_j(P)$  occur in complex conjugate ( $cc$ ) pairs. With the expansion  $u(P) = \sum_{j=1}^{2N} a_j \varphi_j(P)$ , where  $\varphi_{N+j} = \bar{\varphi}_j$ , Galerkin discretization of (2) leads to the quadratic eigenvalue problem

$$\sum_{l=1}^{2N} [-\omega^2 M_{kl} + i\omega G_{kl} + K_{kl}] a_l = 0, \quad k = 1, 2, \dots, 2N \quad (3)$$

$$M_{kl} = (M \varphi_l, \varphi_k) = \bar{M}_{lk}, \quad G_{kl} = (G \varphi_l, \varphi_k) = -\bar{G}_{lk},$$

$$K_{kl} = (K \varphi_l, \varphi_k) = \bar{K}_{lk} \quad (4)$$

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS. Manuscript received May 1999; revised March 2000; Associate Technical Editor: K. W. Wang.

Alternatively, in accordance with Meirovitch [7], D'Eleuterio and Hughes [8], and Wickert and Mote [9], (1) is written in state space form as

$$\mathbf{A}\mathbf{z}_1 + \mathbf{B}\mathbf{z}_2 = 0 \quad (5)$$

$$\mathbf{A} = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}, \quad \mathbf{z} = \begin{Bmatrix} w_l \\ w \end{Bmatrix} \quad (6)$$

With the inner product  $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle = \int_P \mathbf{z}_1^T \bar{\mathbf{z}}_2 dP$  defined on the state space,  $\mathbf{A}$  is self-adjoint and  $\mathbf{B}$  is skew-self-adjoint. The eigenvalue problem associated with (5) can be formulated in terms of two self-adjoint operator matrices  $\mathbf{C}$  and  $\mathbf{A}$  [10]

$$\mathbf{C} = \begin{bmatrix} iG & K \\ K & 0 \end{bmatrix}, \quad \mathbf{v} = \begin{Bmatrix} \omega u \\ u \end{Bmatrix} \quad (7)$$

$$(\mathbf{C} - \omega \mathbf{A})\mathbf{v} = \mathbf{0} \quad (8)$$

This is referred to as the *state space* form in this work. The second of (7) relates the state and configuration space eigenfunctions. With the expansion  $\mathbf{v} = \sum_{j=1}^{4N} b_j \Phi_j(x)$ , where  $\Phi_j(x) = \{\beta_j \varphi_j, \varphi_j\}^T$  are the state space eigenfunctions of a related gyroscopic system and  $\Phi_{2N+j}(x) = \{-\beta_j \bar{\varphi}_j, \bar{\varphi}_j\}^T$ , Galerkin discretization of (8) gives

$$\sum_{l=1}^{4N} (C_{kl} - \omega A_{kl}) b_l = 0, \quad k = 1, 2, \dots, 4N \quad (9)$$

$$C_{kl} = \langle \mathbf{C}\Phi_l, \Phi_k \rangle = \bar{C}_{lk}, \quad A_{kl} = \langle \mathbf{A}\Phi_l, \Phi_k \rangle = \bar{A}_{lk}$$

Note that (3) is quadratic in  $\omega$  and (9) is linear in  $\omega$ , so twice as many terms must be used with (9) to equalize the computational effort in comparing the two approaches. Expansion of (9) gives

$$[-\omega\{\beta_k \beta_l M_{kl} + K_{kl}\} + \{i\beta_k \beta_l G_{kl} + (\beta_k + \beta_l)K_{kl}\}] b_l = 0 \quad (10)$$

where  $M_{kl}$ ,  $G_{kl}$  and  $K_{kl}$  are defined in (4). (The alternative basis function choice  $\Phi_j = \{\omega \varphi_j, \varphi_j\}^T$  where  $\omega$  is an *unspecified* parameter corresponding to the desired eigenvalue [5] gives  $-\omega^3 M_{kl} b_l + i\omega^2 G_{kl} b_l + \omega K_{kl} b_l = 0$  which is identical to (3) for nontrivial  $\omega$ . Thus, this choice is not considered further.)

The configuration and state space discretizations (3) and (10) are *not* equivalent when complex eigenfunctions are used as basis functions. The limitations of each are discussed in the following. Using real basis functions (such as the stationary system eigenfunctions) and letting the indices in the configuration and state space expansions go to  $N$  and  $2N$ , (3) and (10) can be shown to predict the same eigenvalues.

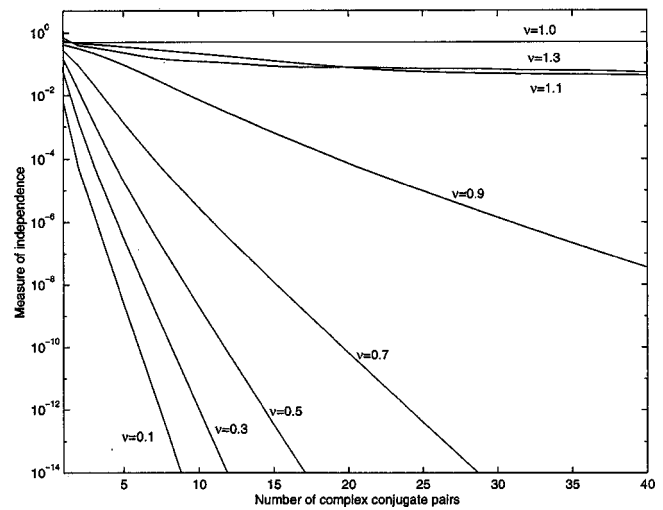
The configuration space suffers numerical problems when many terms are used because the gyroscopic system eigenfunctions are linearly dependent in configuration space as  $N \rightarrow \infty$ . This causes singularity of the mass matrix in (3) and leads to numerical errors and incorrect eigenvalue predictions. As an example, we examine the linear independence of the set of eigenfunctions of an axially moving string with fixed ends for which

$$\beta_n = n\pi(1 - \nu^2), \quad \varphi_n(x) = \sin(n\pi x) e^{in\pi\nu x} \quad (11)$$

Linear independence of a set of functions  $\varphi_1, \varphi_2, \dots, \varphi_r$  is measured by the minimum of the quadratic form in  $r$  variables  $a_1, a_2, \dots, a_r$

$$K(\mathbf{a}, \mathbf{a}) = \left\| \sum_{j=1}^r a_j \varphi_j \right\|^2 = \sum_{i,k=1}^r (\varphi_i, \varphi_k) a_i a_k \quad (12)$$

subject to the restriction  $\sum_{i=1}^r a_i^2 = 1$ . The lowest eigenvalue of  $K(\mathbf{a}, \mathbf{a})$  is called the *measure of independence*, and the functions approach linear dependence as the measure of independence approaches zero [11]. Figure 1 shows the measure of independence versus the number of *cc* pairs of moving string eigenfunctions. At any subcritical speed  $\nu < 1$ , the measure of independence ap-



**Fig. 1 Measure of independence of sets of complex conjugate moving string eigenfunction pairs for varying speeds  $\nu$**

proaches zero with increasing number of eigenfunctions. For low speeds  $0 < \nu \ll 1$ , the complex eigenfunctions have small imaginary part and differ little from the independent, real, stationary ( $\nu = 0$ ) eigenfunctions. There are twice as many complex eigenfunctions compared to the stationary system, however, explaining the rapid decrease in the measure of independence for low speeds. Near the critical speed, the measure of independence decreases very slowly. The exact number of terms that may be included in the expansion before encountering numerical errors hence depends on the speed, with *more* terms being allowed at *higher* speeds. The measure of independence is constant at  $\nu = 1$  where  $\phi_n(x) = i/2(1 - e^{i2n\pi x})$ , meaning that the eigenfunctions are linearly independent at the critical speed (Fig. 1). In this regard, note that any continuous function satisfying the fixed boundary conditions is an eigenfunction (with zero eigenvalue) for  $\nu = 1$ . At supercritical speeds, the eigenfunctions are also linearly independent. The state space basis functions in (9) are the eigenfunctions of a self-adjoint eigenvalue problem (that is, (8) for a related system) and are linearly *independent*; an arbitrary number of terms may be included to achieve the desired accuracy.

The configuration and state space discretizations have different convergence properties. Galerkin's method guarantees the weak convergence of the configuration space discretization in (3) such that

$$\lim_{N \rightarrow \infty} \sum_{j=1}^{2N} a_j^{(n)} \varphi_j = u_n \quad (13)$$

where  $(\omega_n, u_n)$  is the exact  $n$ th eigensolution pair of (2). Similarly, convergence of the state space discretization in (9) is assured such that  $\lim_{N \rightarrow \infty} \sum_{j=1}^{4N} b_j^{(n)} \Phi_j(x) = \mathbf{v}_n$ , or

$$\lim_{N \rightarrow \infty} \sum_{j=1}^{4N} b_j^{(n)} \beta_j \varphi_j = \omega_n u_n \quad (14)$$

$$\lim_{N \rightarrow \infty} \sum_{j=1}^{4N} b_j^{(n)} \varphi_j = u_n \quad (15)$$

The additional condition (14), which represents convergence of the velocity, differs from (15) because the  $\beta_j$ , which are the eigenvalues of the related system from which the basis functions are drawn, differ from the true eigenvalue  $\omega_n$ . Thus, one expects the configuration space discretization to converge more rapidly because the  $a_j^{(n)}$  are selected by Galerkin's method to satisfy only (13), while the  $b_j^{(n)}$  are attempting to satisfy (14) in addition to

(15) (which is the same as (13)). Results presented in the examples confirm this differing convergence rate. Essentially, the state space form imposes an approximation of the eigenvalue ( $\beta_j$ ) in addition to the usual series approximation of the eigenfunction. Whereas the configuration space discretization performance depends only on the ability of the series expansion to represent the true eigenfunction, the state space performance also depends on how close the (specified)  $\beta_j$  are to the (unknown)  $\omega_n$ . Indeed, in the results to follow, the state space discretization converges more slowly as the difference between the  $\beta_j$  and  $\omega_n$  increases. As an extreme example, consider the use of moving string eigenfunctions as a discretization basis for a related system (e.g., a moving string on elastic foundation or a moving beam). For  $\nu=1, \beta_j=0$  for all  $j$  and the state space discretization predicts vanishing eigenvalues (critical speeds) for any system (see (10)). This is clearly in error for moving beam problems where  $\nu=1$  is not a critical speed. Configuration space discretization does not have this problem.

The numerical problems caused by linear dependence of the configuration space eigenfunctions can be viewed through the convergence criteria (13)–(15). For large  $N$ , the  $a_j^{(n)}$  of (13) are not uniquely determined because of the linear dependence of the  $\varphi_j$ , and this corresponds to numerical problems. For the same reason, the  $b_j^{(n)}$  cannot be determined solely from (15) for large  $N$ . The additional condition (14), however, removes the indeterminacy and the  $b_j^{(n)}$  are uniquely calculable.

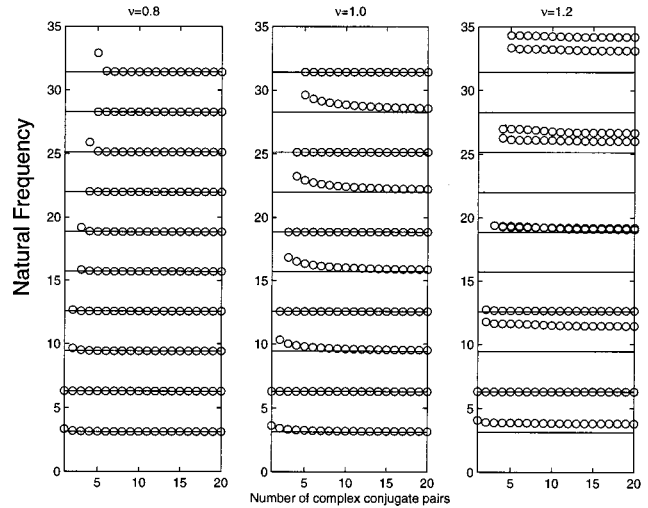
In summary, subcritical speed convergence is more rapid with the configuration space discretization, but numerical errors can result if many terms are required. While subcritical speed convergence is slower with state space discretization, it does not suffer from numerical problems.

A valuable feature of the typical Galerkin discretization is that the eigenvalue predictions converge monotonically from above. This is not the case for state space discretization as seen from the Rayleigh quotient for subcritical speeds given by [10]

$$R(\mathbf{x}) = \frac{\langle \mathbf{C}\mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle} \quad (16)$$

Because  $\mathbf{A}$  and  $\mathbf{C}$  are self-adjoint, the Galerkin discretization (9) is equivalent to Ritz discretization of (16).  $R(\mathbf{x})$  has no lower or upper bound at subcritical speeds as demonstrated by  $R(\mathbf{v}_n) = \omega_n$ ,  $R(\bar{\mathbf{v}}_n) = -\omega_n$ , and  $\lim_{n \rightarrow \infty} \omega_n = \infty$ . Consequently, proofs for convergence from above do not hold for state space discretization. Indeed, results to follow show eigenvalue predictions below exact values. In contrast, for the configuration space form, upper bound approximations are assured for subcritical speeds. This can be seen by writing (2) as  $(i\omega G + K)u = \omega^2 Mu$  and observing that the operators  $(i\omega G + K)$  and  $M$  are self-adjoint and positive-definite for  $\nu < 1$ , guaranteeing convergence from above [1].

The foregoing emphasizes subcritical speed discretizations, but additional concerns arise for supercritical speeds. In fact, none of the possible combinations of a) moving string versus stationary string basis, and b) configuration versus state space discretization, yield acceptable results for moving string problems at supercritical speeds. The moving system basis also fails for moving beam problems, while the stationary system basis performs well. The key problem, discussed in the examples to follow, is that the moving system basis appears to form an incomplete set at supercritical speeds. To further examine this issue, we discretize the stationary string problem in configuration space using moving string eigenfunctions. Figure 2 shows the results for three different speeds. At  $\nu=0.8$ , all modes have converged. At  $\nu=1$ , all even modes converge rapidly but odd modes converge slowly. At supercritical speeds (such as  $\nu=1.2$ ), the eigenvalues converge to incorrect values for several modes. Consequently, this basis shows little



**Fig. 2 Configuration space discretization of a stationary string using moving string eigenfunctions. (○) denote discretization results and solid lines denote the exact eigenvalues of the stationary string.**

promise for any supercritical speed eigenvalue problems, including those for vibration about nontrivial equilibria of nonlinear models.

For axially moving media problems, three choices for discretization basis functions are: 1) eigenfunctions of the corresponding stationary system, 2) configuration space eigenfunctions of a related moving media system, and 3) state space eigenfunctions of a related moving media system. The following sections contrast these options for moving string and beam examples for which exact solutions are available to assess discretization performance.

### Example 1: Axially Moving String on Elastic Foundation.

The linearized, nondimensional equation of motion for an axially moving string supported by a distributed elastic foundation is

$$w_{tt} + 2\nu w_{xt} - (1 - \nu^2)w_{xx} + \kappa w = 0 \quad (17)$$

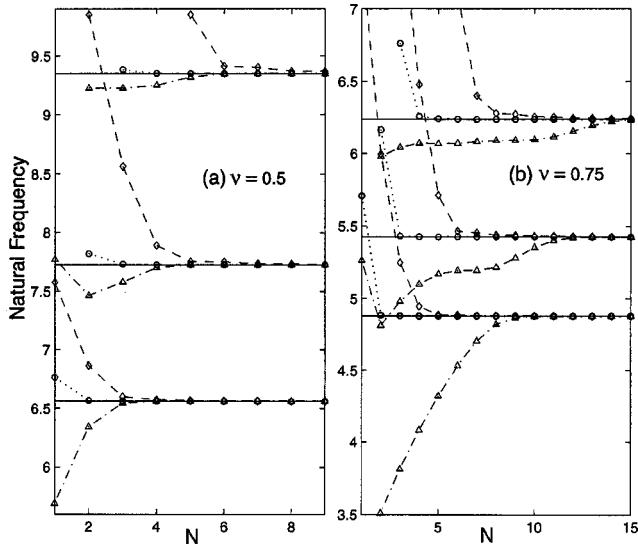
where  $\kappa$  is the stiffness density of the foundation. This problem has been studied for subcritical [12] and supercritical speeds [13]. The exact eigensolutions are

$$\omega_n^2 = -\kappa(\nu^2 - 1) + n^2\pi^2(\nu^2 - 1)^2 \quad (18)$$

$$u_n = \sin(n\pi x)e^{(-i\nu\omega_n/\nu^2-1)x}, \quad n = 1, 2, \dots \quad (19)$$

We apply both configuration space and state space discretizations to this problem using the eigenfunctions of the axially moving string as basis functions (see (11)). With configuration space discretization, the number of basis functions at which linear dependence causes numerical problems increases with speed (see Fig. 1). For example, with  $\kappa=50$ , one may include up to 8  $cc$  pairs at  $\nu=0.05$ , up to 21  $cc$  pairs at  $\nu=0.5$  and over 100 pairs at  $\nu=0.9$ .

Figure 3 compares the rates of convergence for the three approaches mentioned above, for a relatively high stiffness of  $\kappa = 50$  and two subcritical speeds,  $\nu=0.5$  and  $\nu=0.75$ . Considering first the moving string basis functions, the configuration space form clearly converges faster than the state space form. Note that the state space form converges slower at  $\nu=0.75$  than at  $\nu=0.5$ . This is because higher speeds, higher stiffnesses and lower modes are the conditions under which the eigenvalues  $\beta_j$  in the basis functions differ the most from the exact system eigenvalues (see (18)). Note also that state space predictions converge nonmonotonically from below. Configuration space predictions, on the



**Fig. 3** Discretization of an axially moving string on elastic foundation with  $\kappa=50$ . Figures (a) and (b) show results for  $\nu=0.5$  and  $\nu=0.75$ , respectively. ( $\diamond$ )  $-N$  terms of stationary string eigenfunctions, ( $\circ$ )  $-$ configuration space form using  $N$  complex conjugate moving string eigenfunction pairs, and ( $\triangle$ )  $-$ state space form with  $2N$  complex conjugate moving string eigenfunction pairs. Horizontal solid lines denote the lowest three exact eigenvalues.

other hand, always converge monotonically from above. For low speeds, low stiffnesses, and higher modes, the configuration and state space forms yield comparable results.

It is interesting to compare discretization using stationary string modes with the above two approaches using moving string modes. An analytical estimate for the minimum number of stationary string modes required for accurate results is obtained by Fourier expansion of the exact eigenfunctions in (19). With the expansion  $u_n = \sum_{j=1}^{\infty} c_j \sin(j\pi x)$ , the Fourier coefficients of  $u_n$  are

$$c_j = \frac{2in j \alpha_n [e^{i\alpha_n (-1)^{n+j} - 1}]}{[(n\pi - \alpha_n)^2 - j^2 \pi^2][(n\pi + \alpha_n)^2 - j^2 \pi^2]},$$

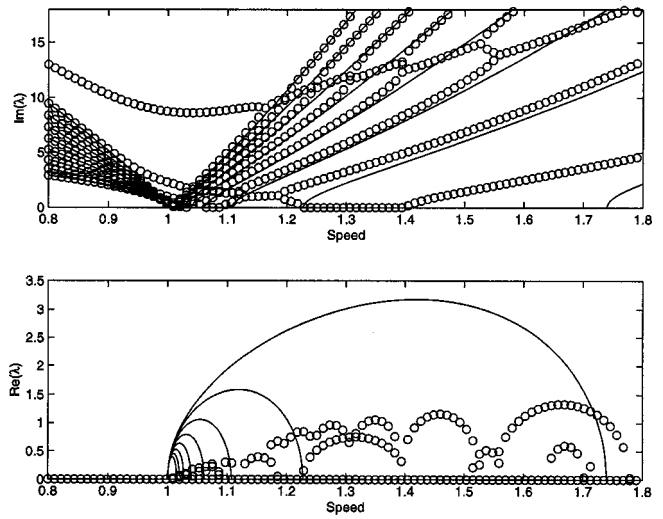
$$\alpha_n = \frac{\nu}{\nu^2 - 1} |\omega_n| \quad (20)$$

Here  $i = \sqrt{-1}$ . For given  $n$  and  $\nu$ , the denominator of (20) vanishes for

$$j_{1,2} = n \left[ 1 \pm \nu \sqrt{1 + \frac{\kappa}{n^2 \pi^2 (1 - \nu^2)}} \right] \quad (21)$$

The indices  $j_{1,2}$  identify the modes that have major contributions and must be included in the expansion for accurate results. Hence, the minimum number of terms required is given by  $N = \text{integer part} [\max(j_{1,2}) + 2]$ , where an extra term is added to ensure that all major modes are included. This expression shows that the number of terms required increases with foundation stiffness and speed, as expected. For instance, for  $\nu=0.75$  and  $\kappa=50$ , one calculates  $N=5$  for the first mode ( $n=1$ ) and  $N=6$  for the second mode. Figure 3 shows that these are the minimum values of  $N$  where reasonable convergence is achieved. A similar analysis was carried out for the axially moving string without elastic foundation by Lengoc and McCallion [14].

None of the discretization approaches considered here yield correct (or even reasonable) eigenvalue estimates at supercritical speeds. Parker [4] showed that for moving string systems linearized about the trivial equilibrium, spatial discretization at supercritical speeds ( $\nu > 1$ ) using the stationary system eigenfunctions yields inaccurate and misleading results, including spurious flutter



**Fig. 4** Configuration space discretization of an axially moving string on elastic foundation ( $\kappa=20$ ) showing incorrect flutter predictions. ( $\circ$ ) denote eight  $cc$  pairs of moving string eigenfunctions and solid lines denote exact eigenvalues.

predictions, regardless of the number of terms used. Configuration space discretization using moving string eigenfunctions also predicts seriously inaccurate results (Fig. 4). Recall that this moving string basis appears to be incomplete for supercritical speeds. In addition to poor convergence, flutter instability that does not exist in the continuum model is incorrectly predicted, *regardless of the number of terms used*. The state space discretization is even more striking, predicting *infinite* supercritical eigenvalues. To show this analytically, the single mode expansion  $\mathbf{v} = b_n \Phi_n + b_{-n} \Phi_{-n}$ , where  $\Phi_{\pm n}$  are the  $n$ th mode  $cc$  moving string eigenfunctions, in (10) yields the eigenvalue estimate

$$\omega^2 = \frac{4\delta^4 \nu^2 [n\pi\delta + \kappa]^2}{4\delta^2 \nu^2 [n\pi\delta + \kappa/2]^2 - [\kappa^2 \sin^2(n\pi\nu)]} \quad (22)$$

where  $\delta = n\pi(1 - \nu^2)$ . The denominator of the above expression vanishes at some supercritical speed, predicting an infinite eigenvalue. This error persists when more terms are included in the expansion. This can also be seen from the Rayleigh quotient in (16) where the operator  $\mathbf{A}$  is no longer positive definite at supercritical speeds.

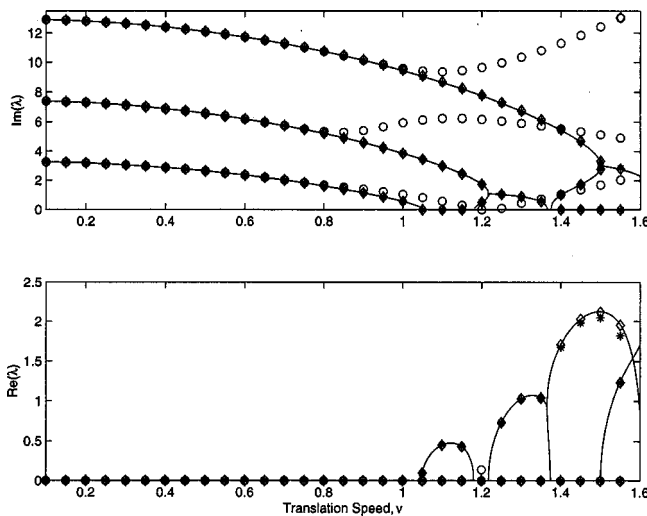
All of the above erroneous supercritical predictions were also observed for the case of multiple, discrete elastic supports.

**Example 2: Axially Moving Beam.** The nondimensional equation of motion for an axially moving, tensioned, Euler-Bernoulli beam is [15]

$$w_{tt} + 2\nu w_{xt} - (1 - \nu^2)w_{xx} + \gamma^2 w_{xxxx} = 0 \quad (23)$$

where  $\gamma = EI/TL^2$  represents the flexural stiffness of the beam. Exact natural frequencies are obtained by numerical solution of the frequency equation [1].

Figure 5 compares discretization results with exact eigenvalues. Stationary beam eigenfunctions yield excellent convergence at subcritical *and* supercritical speeds. These are the superior choice. The moving string eigenfunctions are admissible functions for pinned end conditions and, because they capture the eigenfunction speed dependence, might be expected to yield better approximations than stationary beam eigenfunctions at high speeds. Surprisingly, this is not the case. Configuration space form yields satisfactory results only for low speeds ( $\nu < 1$ ) (Fig. 5). The error increases sharply for  $\nu \geq 1$ ; in fact, results converge to incorrect answers, consistent with the apparent loss of completeness discussed earlier.



**Fig. 5 Discretization of a simply supported, axially moving, tensioned beam ( $\gamma=0.1$ ). ( $\diamond$ ) - six terms of stationary beam eigenfunctions, ( $\circ$ ) - six cc pairs of moving string eigenfunctions, and ( $*$ ) - six cc pairs of modified moving string eigenfunctions with  $\alpha=2$ . Solid lines denote exact eigenvalues.**

Numerical experiments suggested use of the modified moving string eigenfunctions  $\psi_n(x) = \sin(n\pi x)e^{in\pi x/\alpha}$ . Note that the stationary beam and moving string eigenfunctions are recovered from  $\psi_n$  as  $\alpha$  approaches  $\infty$  and 1, respectively. Figure 5 shows the markedly improved convergence for  $\alpha=2$ . In general,  $\alpha > 1$  extends the range for which convergence is possible to  $\nu = \alpha$ . For  $\nu \geq \alpha$ , however, results converge to incorrect answers from loss of completeness.

State space discretization using moving string eigenfunctions fails for the moving beam because zero eigenvalues are incorrectly predicted for all modes at  $\nu=1$ , as discussed earlier. Also, infinite eigenvalues are predicted at supercritical speeds as for the moving string.

## Summary and Conclusions

1 The configuration space and state space discretization approaches using complex eigenfunctions are *not* equivalent. Configuration space form has superior convergence properties but suffers limitations arising from linear dependence of complex eigenfunctions. The state space form, though free of this problem, suffers due to its strong dependence on the eigenvalues of the related system from which the basis functions are drawn. This results in poor convergence in some cases. Also, upper bound approximations are not guaranteed in the state space discretization.

2 The moving string eigenfunctions approach linear dependence at subcritical speeds as the number of eigenfunctions grows. This causes numerical problems if many terms are needed. While they are linearly independent at supercritical speeds ( $\nu > 1$ ), they appear to form an incomplete set and are unsuitable as a discretization basis at supercritical speeds. This conclusion applies when examining vibrations about trivial and nontrivial supercritical equilibria.

3 At supercritical speeds, spatial discretization of the axially

moving string on elastic foundation problem leads to inaccurate and misleading results regardless of whether stationary or moving system basis functions are used. For the trivial equilibrium (and possibly nontrivial equilibria) of the supercritical *string*, there appears to be no reliable discretization approach to obtain eigenvalue predictions.

4 Stationary system basis functions yield consistently accurate subcritical speed results for all problems considered in this work. For dispersive gyroscopic systems such as the moving beam, spinning disk, and spinning disk-spindle system (Parker and Sathe, 1999), stationary system bases also yield excellent supercritical speed results. Although stationary system basis functions require more terms than moving system basis functions, they have no linear dependence concerns and are computationally efficient because the mass, gyroscopic and stiffness matrices are computed only once and used for all speeds. Hence these are the most reliable basis functions to use when the interest lies in *obtaining the natural frequencies and mode shapes*. When discretization is used in nonlinear and time-varying problems, however, the superior convergence of the moving system basis has great advantage. In these cases, single-mode analyses yield excellent results and admit analytical perturbation results where only numerical results are possible with multi-mode discretizations using the stationary system basis [15–18].

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