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STRUCTURED EIGENSOLUTION PROPERTIES OF PLANETARY GEARS WITH ELASTICALLY DEFORMABLE RING GEARS

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ABSTRACT

The distinctive modal properties of equally spaced planetary gears with elastic ring gears are studied through perturbation and a candidate mode method. All eigenfunctions fall into one of four mode types whose structured properties are derived analytically. Two perturbations are used to obtain closed-form expressions of all the eigenfunctions. In the Discrete Planetary Perturbation (DPP), the unperturbed system is a discrete planetary gear with a rigid ring. The stiffness of the ring is perturbed from infinite to a finite number. In the Elastic Ring Perturbation (ERP), the unperturbed system is an elastic ring supported by the ring-planet mesh springs; the sun, planet and carrier motions are treated as small perturbations. A subsequent candidate mode method analysis proves the perturbation results and removes any reliance on perturbation parameters being small. All vibration modes are classified into rotational, translational, planet and purely ring modes. The well defined properties of each type of mode are analytically determined. All modal properties are verified numerically.

INTRODUCTION

Planetary gears are widely used in automotive and aerospace transmissions due to advantages such as compactness, high torque/weight ratio, low bearing load and high transmission ratio. In practical systems where planetary gear vibration is a key concern, ring gear elastic deformation is significant. This is especially true for planetary gears with thin rims, including those used in aerospace applications. The free vibration of planetary gears with equally spaced planets has typically been studied by treating all the planetary gear components as rigid bodies [1-7]. Lin and Parker [5] established a lumped parameter model that includes both transverse and torsional motion. The modal properties were obtained analytically, and the vibration modes are classified into rotational, translational and planet modes. In the present paper, these modes are called *discrete* rotational, translational and planet modes. Lin and Parker used this discrete model to study natural frequency and vibration mode sensitivity [8], natural frequency veering [9], and parametric instability caused by changing contact conditions at the multiple tooth meshes [10].

The influence of gear rim flexibility on static and dynamic behavior of planetary gears has been studied by a few researchers. Kahraman and Vijayakar [11] investigated the impact of ring gear rim flexibility on gear stresses and planet load sharing under static conditions using a deformable-body planetary gear model. The study indicates that reducing rim thickness minimizes the adverse effects of gear and carrier manufacturing errors and improves the planet load sharing. Kahraman et al. [12] studied the effect of gear rim flexibility on dynamic behavior of planetary gears using the finite element method, exposing the importance of ring deformation for practical systems.

This study analytically addresses the dynamics of planetary gears having elastic ring gears. An elastic-discrete model is developed, where the ring gear is modeled as an elastic body while all other gears are represented as rigid bodies. Modal properties are derived in detail using eigenvalue perturbation and a candidate mode method. Two unperturbed systems are considered to form a complete representation for all modes. This yields

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closed-form expressions for all the eigenfunctions and a systematic characterization of planetary gears' highly structured modal properties. All vibration modes are classified in detail into four different types according to their unique characteristics. These perturbation results are proved by a mathematically rigorous approach where vibration modes having the form revealed by perturbation are assumed and then shown to satisfy all equations of the elastic-discrete eigenvalue problem. This builds a base for subsequent studies such as dynamic response, parametric instability, and contact loss nonlinearity, all of which commonly use modal expansion methods.

This work presents that published in the ASME paper [14].

NOMENCLATURE

Superscript 0 and 1 of $\mathbf{a}, \mathbf{q}, v$denote the unperturbed and the first order perturbation eigenfunctions, respectively. Subscripts c, r, s, n denote the carrier, ring, sun and the n th planet.

α_r	Ring-planet pressure angle
k_{bend}	Ring bending stiffness
α_s	Sun-planet pressure angle
k_{rp}	Ring-planet mesh stiffness
ψ_n	Location of the n-th planet
k_{sp}	Sun-planet mesh stiffness
ψ_{rn}	$= \psi_n + \alpha_r$
$k_{rbs}\, k_{rus}$	Radial, tangential distributed ring elastic foundation stiffnesses
ψ_{sn}	$= \psi_n - \alpha_s$
I_j	Mass moment of inertia for the ring, carrier and sun, $j = r, c, s$
ν	Poisson's ratio
m_j	Mass of the ring, carrier and sun, $j = r, c, s$
	Mass density per unit length
r_j	Base radius for the ring and sun, $j = r, c, s$
	Young's modulus
r_1, r_2	Inner, outer radii of the ring gear
	Area moment of inertia
u, w	Ring tangential, radial deflections
N	Number of planets
v	Ring elastic tangential deflection
R	Neutral radius of the ring gear
x_j, y_j, u_j	Translational and rotational displacements of the ring, sun and carrier, $j = r, c, s$
k_j, k_{ju}	Translational and rotational stiffness of supports/bearing for the carrier and sun, $j = c, s$
ξ_n, η_n, u_n	Radial, tangential, and rotational displacements of the n^{th} planet

MODELING AND EQUATIONS OF MOTION

An elastic-discrete model of a planetary gear is shown in Figure 1. All gear meshes are represented by linear springs. The sun, carrier and planets are considered as rigid bodies, while the ring gear is modeled as a thin elastic body. The bearings and supports of the sun, carrier and planets are modeled as two perpendicular springs of equal stiffness. The bearings and supports of the ring gear are represented as an elastic foundation with uniform radial and tangential distributed stiffnesses per unit length k_{rbs} and k_{rus}, respectively. The planets are identical and equally spaced. All ring-planet mesh stiffnesses are equal (k_{rp}), and all sun-planet mesh stiffnesses are equal (k_{sp}), where k_{rp} and k_{sp} are averages over a mesh cycle. The angular speeds are assumed to be small, so gyroscopic effects are neglected.

The coordinates are shown in Figure 1. The deformation of the sun and carrier $\mathbf{p}_j = [x_j, y_j, u_j]^T$, $j = s, c$ are described relative to the fixed basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$; the tangential displacement of the ring is $u(\theta, t)$; the ring radial deflection is determined from the inextensibility condition$w = -\partial u/\partial\theta$...[13] ; and, the deflections of the planets are $\mathbf{p}_n = [\xi_n, \eta_n, u_n]^T$, $n = 1, \cdots, N$. The symbol u_j denotes rotational (or tangential) deflection (rotation in radians times the gear base radii r_s, r_r, r_p or radius of the carrier r_c).

The equations of motion for the sun and carrier are the same as those in the discrete model [5], while the equations of motion for the ring and planets change. The equation of motion for the elastic ring gear is [13]

$$M_e\ddot{u} + k_{bend}L_1 u + k_{rp}L_2 u + \\ k_{rp}\sum_{n=1}^{N} L_3^n(\xi_n \sin\alpha_r - \eta_n\cos\alpha_r - u_n) = 0 \tag{1}$$

$$M_e = \rho R(1 - \frac{\partial^2}{\partial\theta^2})$$

$$k_{bend} = \frac{EJ}{R^3(1-\nu^2)}$$

$$L_1 = -(\frac{\partial^6}{\partial\theta^6} + 2\frac{\partial^4}{\partial\theta^4} + \frac{\partial^2}{\partial\theta^2})$$

$$L_2 = -\sum_{n=1}^{N}[(\sin^2\alpha_r\tfrac{\partial^2}{\partial\theta^2} - \cos^2\alpha_r)\delta(\theta-\psi_n) + (\sin\alpha_r\tfrac{\partial}{\partial\theta} + \cos\alpha_r) \sin\alpha_r\tfrac{\partial\delta(\theta-\psi_n)}{\partial\theta}(\sin\alpha_r\tfrac{\partial}{\partial\theta} + \cos\alpha_r)\sin\alpha_r\tfrac{\partial\delta(\theta-\psi_n)}{\partial\theta}] + (k_{rus}R - k_{rbs}R\tfrac{\partial^2}{\partial\theta^2})/k_{rp},$$

$$L_3^n = \cos\alpha_r\delta(\theta-\psi_n) - \sin\alpha_r\frac{\partial\delta(\theta-\psi_n)}{\partial\theta}, \qquad (2)$$

where k_{bend} is the ring bending stiffness (see Appendix A for nomenclature). L_1, L_2 and L_3^n are dimensionless operators. The first two terms of (1) represent the in-plane vibration of a free ring; the last two terms incorporate the effects of gear meshes and elastic supports.

Separation of the ring rigid body motions from the elastic deformation $v(\theta,t)$ is achieved with the expansion

$$\begin{aligned} u(\theta,t) &= v(\theta,t) + U_1(t)e^{i\theta} + U_{-1}(t)e^{-i\theta} + U_0(t) \\ &= \sum_{m=\pm 2}^{\pm\infty} V_m(t)e^{im\theta} + U_1(t)e^{i\theta} + U_{-1}(t)e^{-i\theta} + U_0(t). \end{aligned} \qquad (3)$$

Thus, v is orthogonal to the rigid body motions

$$\int_0^{2\pi} v\,d\theta = 0, \quad \int_0^{2\pi} v e^{i\theta}d\theta = 0, \quad \int_0^{2\pi} v e^{-i\theta}d\theta = 0. \qquad (4)$$

Substituting (3) into (1) and forming the inner product of the result with $e^{im\theta}$ yields the discretized equations of motion. Comparison of the equations for the rigid ring motions U_1, U_{-1}, U_0 to the equations of motion for a rigid ring planetary gear model with variables $\mathbf{p}_r = (x_r,\ y_r, u_r)^T$ [5] yields the relations

$$x_r = -i(U_1 - U_{-1}), \quad y_r = U_1 + U_{-1}, \quad u_r = U_0\cos\alpha_r, \quad I_r = m_r R^2. \qquad (5)$$

The true moment of inertia expression for a ring is $I_{r,true} = m_r(r_2^2 + r_1^2)/2$, where r_1 and r_2 are the inner and outer radii of the ring gear, respectively. The difference between I_r and $I_{r,true}$ is small when the ring is thin.

We introduce the following dimensionless quantities

$$\begin{aligned} &\tilde{v} = \frac{v}{R}, \quad \tau = \frac{t}{T}, \quad T = \sqrt{\frac{m_r}{k_{rp}}}, \quad \tilde{k}_i = \frac{k_i}{k_{rp}}, \\ &\qquad i = r,c,s,p,rp,sp,bend, \\ &\tilde{k}_{rbs} = \frac{k_{rbs}R}{k_{rp}}, \tilde{k}_{rus} = \frac{k_{rus}R}{k_{rp}}, \quad \tilde{m}_j = \frac{m_j}{m_r} \end{aligned} \qquad (6)$$

$$\tilde{I}_j = \frac{I_j}{m_r r_j^2}, \quad j = r,c,s,n. \qquad (7)$$

In what follows the $\sim$ on all variables is omitted, and the equations of motion remain the same except that k_{rp} is replaced by 1, M_e is replaced by $\frac{1}{2\pi}(1-\frac{\partial^2}{\partial\theta^2})$, and $k_{rbs}R$, $k_{rus}R$ are replaced by k_{rbs}, k_{rus}.

The displacement of the whole system is separated into $v(\theta,\tau)$ and $\mathbf{q}(\tau)$. v is the elastic deformation of the ring gear, and $\mathbf{q}$ is a vector of the deflections for the discrete elements including the ring rigid body motions

$$\mathbf{q} = [\underbrace{x_r, y_r, u_r}_{\mathbf{p}_r}, \underbrace{x_c, y_c, u_c}_{\mathbf{p}_c}, \underbrace{x_s, y_s, u_s}_{\mathbf{p}_s}, \underbrace{\xi_1, \eta_1, u_1}_{\mathbf{p}_1}, \cdots, \underbrace{\xi_N, \eta_N, u_N}_{\mathbf{p}_N}]^T \qquad (8)$$

The dimensionless equations of motion and the associated eigenvalue problem in extended operator forms are

$$M\ddot{\mathbf{a}} + K\mathbf{a} = 0, \qquad (9)$$

$$-\omega_i^2 M\mathbf{a}_i + K\mathbf{a}_i = 0, \qquad (10)$$

where $\mathbf{a} = \left[v, \mathbf{q}^T\right]^T$ is referred to as an *extended variable*, ω_i is a natural frequency, and M, K are extended stiffness and inertia operators defined by their action on elements of the space of extended variables according to

$$M\mathbf{a} = \begin{bmatrix} \frac{1}{2\pi}(1-\frac{\partial^2}{\partial\theta^2})v \\ \mathbf{Mq} \end{bmatrix}, \quad K\mathbf{a} = \begin{bmatrix} (k_{bend}L_1 + L_2)v + L_3\mathbf{q} \\ L_4 v + \mathbf{Kq} \end{bmatrix}, \qquad (11)$$

$$L_3\mathbf{q} = \sum_{n=1}^{N}\left[\cos\alpha_r \delta(\theta-\psi_n) - \sin\alpha_r \frac{\partial\delta(\theta-\psi_n)}{\partial\theta}\right]\delta_n, \quad (12)$$

$$L_4 v = \left[\sum_{n=1}^{N}(\mathbf{b}_r^T \chi|_{\theta=\psi_n}), \mathbf{0}, \mathbf{0}, \mathbf{b}_p^T\, \chi|_{\theta=\psi_1}, \cdots, \mathbf{b}_p^T\, \chi|_{\theta=\psi_N}\right]^T$$
$$\chi = \frac{\partial v}{\partial\theta}\sin\alpha_r + v\cos\alpha_r \quad (13)$$

$$\mathbf{b}_r = [-\sin\psi_{rn},\ \cos\psi_{rn},\ 1]^T, \quad \mathbf{b}_p = [\sin\alpha_r,\ -\cos\alpha_r,\ -1]^T, \quad (14)$$

$$\begin{array}{l}\delta_n = -x_r\sin\psi_{rn} + y_r\cos\psi_{rn} + u_r \\ +\xi_n\sin\alpha_r - \eta_n\cos\alpha_r - u_n\end{array}. \quad (15)$$

M and K are self-adjoint with the inner product $< \mathbf{a}_1, \mathbf{a}_2 >= \int_0^{2\pi} v_1\bar{v}_2 d\theta + \mathbf{q}_1^T\bar{\mathbf{q}}_2$, where an overbar denotes complex conjugate. $\mathbf{M}$ and $\mathbf{K}$(see Appendix B for details) are the dimensionless mass and stiffness matrices for planetary gears based on a discrete model. Their dimensional forms are identical to the mass and stiffness matrices in ....[5] with the only difference in $\mathbf{M}_r$ and $\mathbf{K}_{rb}$ as

$$\begin{array}{r}\mathbf{M}_r = diag(1,\,1,1/\cos^2\alpha_r), \\ \mathbf{K}_{rb} = \pi diag(k_{rbs}+k_{rus}, k_{rbs}+k_{rus}, 2k_{rus}/\cos^2\alpha_r)\end{array} \quad (16)$$

Expansion of (10) into $N+4$ groups of equations associated with the individual components yields

$$-\frac{\omega_i^2}{2\pi}(1-\frac{\partial^2}{\partial\theta^2})v + k_{bend}L_1 v + L_2 v + L_3\mathbf{q} = 0, \quad (17)$$

$$\begin{array}{l}-\omega_i^2\mathbf{M}_r\mathbf{p}_r + (\mathbf{K}_{rb} + \sum_n \mathbf{K}_{r1}^n)\mathbf{p}_r + \sum_n \mathbf{K}_{r2}^n\mathbf{p}_n \\ +\sum_n(\mathbf{b}_r\,\chi|_{\theta=\psi_n}) = 0\end{array}, \quad (18)$$

$$-\omega_i^2\mathbf{M}_c\mathbf{p}_c + (\mathbf{K}_{cb} + \sum_n \mathbf{K}_{c1}^n)\mathbf{p}_c + \sum_n \mathbf{K}_{c2}^n\mathbf{p}_n = \mathbf{0}, \quad (19)$$

$$-\omega_i^2\mathbf{M}_s\mathbf{p}_s + (\mathbf{K}_{sb} + \sum_n \mathbf{K}_{s1}^n)\mathbf{p}_s + \sum_n \mathbf{K}_{s2}^n\mathbf{p}_n = \mathbf{0}, \quad (20)$$

$$\begin{array}{l}-\omega_i^2\mathbf{M}_p\mathbf{p}_n + (\mathbf{K}_{c2}^n)^T\mathbf{p}_c + (\mathbf{K}_{r2}^n)^T\mathbf{p}_r + (\mathbf{K}_{s2}^n)^T\mathbf{p}_s \\ +\mathbf{K}_{pp}\mathbf{p}_n + \mathbf{b}_p\,\chi|_{\theta=\psi_n} = \mathbf{0}\end{array}, \quad n = 1, \cdots, N. \quad (21)$$

Equation (10) is cast entirely in discrete form with modal expansion of v as

$$v(\theta,\tau) = \sum_{m=\pm 2}^{\pm JN} V_m(\tau)e^{im\theta}, \quad (22)$$

where $J \geq 1$ is an integer. The basis functions $e^{im\theta}$ are complete, (22) converges, and J is arbitrarily large. Thus, the error in (22) can be made as small as desired. No restriction is put on J in what follows, so the findings apply to the continuum ring model without any limitation introduced by the expansion (22).

A discretized model results from substitution of (22) into (17)-(21) and then forming the inner product of (17), (18) and (21) with $e^{ip\theta}$. Numerical experiments on the discretized equations confirm that ring elastic deformation alters the natural frequencies and vibration modes compared to the lumped parameter model and introduces additional natural frequencies associated with modes dominated by ring elastic deformation. The numerical solutions indicate that all vibration modes of this elastic-discrete model are classified into four types: rotational, translational, planet and purely ring modes.

For example, a planetary gear with six equally spaced planets is analyzed with $J = 3$ in (22). The system parameters and the dimensionless natural frequencies are listed in Table 1. The natural frequencies in Table 1 include all four mode types: ω_1, ω_9 and ω_{14} are for rotational modes; $\omega_{2,3}$ and $\omega_{7,8}$ are for translational modes; $\omega_{4,5}$ and $\omega_{10,11}$ are for degenerate planet modes (type 2) and ω_6, ω_{13} are for distinct planet modes (type 3); ω_{12} is for a purely ring mode.

Figure 2a shows the vibration mode of a rotational mode (ω_1). From the numerical simulations, a rotational mode has the following characteristics: (a) The discrete elements $\mathbf{q}$ have the same properties as a discrete rotational mode, where the translations of the sun, carrier, and ring rigid motion are zero, and all planets have identical deflections, (b) The associated natural frequency is distinct, (c) The elastic deformation of the ring contains only jN, $j = 1, 2, \cdots, J$ nodal diameter components.

Figure 2b shows the vibration mode of a translational mode ($\omega_{2,3}$). A translational mode has the following characteristics: (a) The discrete elements $\mathbf{q}$ have the same properties as a discrete translational mode, where the rotations of the sun, carrier, and ring rigid motion are zero, and the deflections of the planets are related by a rotation matrix, (b) The associated natural frequency is repeated with multiplicity two, (c) The elastic deformation of the ring contains only $jN \pm 1$ nodal diameter components, where j is any nonzero integer satisfying $jN \pm 1 \in \{-JN, -JN+1, \cdots, JN\}$ (a condition imposed by the $\pm JN$ limits in (22)).

Planet modes are classified into two sub-types according to the degeneracy of the natural frequencies. For odd N all planet modes are degenerate, as are the majority of planet modes for even N. Degenerate planet modes have the following characteristics: (a) The discrete elements $\mathbf{q}$ have the same properties as a discrete planet mode, where the deflections of the sun, carrier, and ring rigid motion are zero, and the deflections of the planets are scalar multiples of the first planet's deflection, (b) The associated natural frequency is repeated with multiplicity two, (c) Each mode is associated with a particular $s \in \{2,3,\cdots,int(\frac{N-1}{2})\}$. For that particular s, the elastic deformation of the ring contains only $jN \pm s$ nodal diameter components, where j is any integer satisfying $jN \pm s \in \{-JN, -JN+1, \cdots, JN\}$. Figure 2c shows a degenerate planet mode ($\omega_{4,5}$) where the two nodal diameter component is the dominant ring deformation. For even N, the remaining planet modes have distinct natural frequencies. Their discrete elements behave as in (a) above, but their elastic ring deflection contains only $jN + N/2$ nodal diameter components, where j is any integer satisfying $jN \pm N/2 \in \{-JN, \cdots, JN\}$ (see Figure 2d for a distinct planet mode).

Thus, planet modes are classified into $int(\frac{N}{2}) - 1$ subtypes according to the ring nodal diameter components they contain. Planet modes having $jN \pm s$ nodal diameter components are named type s planet modes. Each planet mode belongs to a unique type. For the example where $N = 6$, two types exist: the degenerate planet modes are type 2 ($s = 2, \cdots, int(\frac{N-1}{2})$), and the distinct planet modes are type 3 ($s = \frac{N}{2}$) which only exist for even N. There are no planet modes outside of these two types for $N = 6$. Table 2 summarizes the number of degenerate/distinct planet modes and their types for varying numbers of planets.

Figure 2e shows a purely ring mode (ω_{12}). A purely ring mode has the following characteristics: (a) The discrete elements $\mathbf{q}$ are all zero, (b) The natural frequency is distinct, (c) The elastic deformation of the ring contains only a single nodal diameter component.

In this example ($N = 6$, $J = 3$), $3N + 2JN + 7 = 61$ eigensolutions are obtained numerically: $J + 6 = 9$ rotational modes, $4J + 10 = 22$ translational modes, $(2JN - 7J) + (3N - 9) = 24$ planet modes divided as $2J + 3 = 9$ degenerate pairs and $J + 3 = 6$ distinct modes, and $2J = 6$ purely ring modes.

The remainder of this paper analytically proves that these properties (natural frequency multiplicity, modal properties and the number of each type of mode) hold for general planetary gears.

PERTURBATION ANALYSIS

To find all natural frequencies and vibration modes of the elastic-discrete model of a planetary gear, two perturbations are used for different ranges of parameters. For the chosen nondimensional variables, the stiffness of the ring-planet mesh is always unity ($k_{rp} \equiv 1$). The first perturbation is termed *Discrete Planetary Perturbation* (DPP), with the unperturbed system being a discrete planetary gear having a nearly rigid ring gear where the bending stiffness is $O(1/\varepsilon)$ while the stiffnesses of all remaining meshes/supports are $O(1)$. The small quantity ε is the ring bending compliance. The opposite case of DPP is *Elastic Ring Perturbation* (ERP). In this case the bending stiffness is $O(1)$ and the stiffnesses of the remaining meshes/supports (except $k_{rp} \equiv 1$) are $O(1/\varepsilon)$. The unperturbed system for the ERP is an elastic ring having multiple springs with the elimination of the rigid body motions. The attached springs represent the ring-planet gear meshes. The combined eigensolutions from the DPP and ERP form a complete set of eigensolutions for planetary gears having elastic rings without any redundancy (as proved in a subsequent Candidate Mode Method solution). This process leads to analytical results that mathematically expose the system's highly structured modal properties.

Discrete Planetary Perturbation

In DPP, the ring bending stiffness is much larger than the mesh and bearing stiffnesses. The mesh and bearing stiffnesses are $O(1)$, while the ring bending stiffness $k_{bend} = 1/\varepsilon$, where ε is a small parameter. The eigenvalue problem in extended operator form is

$$-\omega^2 M\mathbf{a} + \hat{K}\mathbf{a} = \begin{bmatrix} \frac{-\omega^2}{2\pi}(1 - \frac{\partial^2}{\partial\theta^2})v \\ -\omega^2 \mathbf{Mq} \end{bmatrix} + \begin{bmatrix} \frac{1}{\varepsilon} L_1 v \\ 0 \end{bmatrix} + \begin{bmatrix} L_2 v + L_3 \mathbf{q} \\ L_4 v + \mathbf{Kq} \end{bmatrix} = \mathbf{0} \tag{23}$$

where $M$and $\hat{K}$ are self-adjoint operators. The eigensolutions of (23) are represented as

$$\mathbf{a} = \mathbf{a}^0 + \varepsilon \mathbf{a}^1 + O(\varepsilon^2), \quad \omega^2 = \omega_0^2 + \varepsilon\mu + O(\varepsilon^2), \quad \mathbf{a}^0 = \begin{bmatrix} v^0 \\ \mathbf{q}^0 \end{bmatrix}, \quad \mathbf{a}^1 = \begin{bmatrix} v^1 \\ \mathbf{q}^1 \end{bmatrix}. \tag{24}$$

Substitution of (24) into (23) gives the perturbation equations. The perturbation equation of order ε^{-1} is $L_1 v^0 = 0$. L_1 is positive definite, giving

$$v^0 = 0. \tag{25}$$

Substitution of (25) into the remaining perturbation equations yields

$$-\omega_0^2 \mathbf{Mq}^0 + \mathbf{Kq}^0 = \mathbf{0}, \tag{26}$$

$$L_1 v^1 = -L_3 \mathbf{q}^0, \quad -\omega_0^2 \mathbf{Mq}^1 + \mathbf{Kq}^1 = \mu \mathbf{Mq}^0 - L_4 v^1. \tag{27}$$

Equation (26) is the eigenvalue problem for a discrete (rigid ring) planetary gear model [5]. From (25) and (26), the unperturbed eigenfunction is

$$\mathbf{a}^0 = \begin{bmatrix} 0 \\ \mathbf{q}^0 \end{bmatrix}. \tag{28}$$

The structured properties of the discrete model unperturbed eigensolutions are proven analytically in ....[5], where the discrete system vibration modes $\mathbf{q}^0$ are classified into rotational, translational and planet modes. In this study, they are called *discrete* rotational, translational and planet modes. In the elastic-discrete model, similar mode types are found; they are called rotational, translational, and planet modes. The different mode types are considered separately.

Common to each mode type, v^1 is solved from the first equation of (27) by expanding v^1 as $v^1 = \sum_{m=\pm 2}^{\pm JN} V_m^1 e^{im\theta}$, multiplying (27) by $e^{-im\theta}$, and integrating from 0 to2π. This yields

$$V_m^1 = -\frac{\cos\alpha_r - im\sin\alpha_r}{2\pi m^2(m^2-1)^2} \sum_{n=1}^{N} \delta_n^0 e^{-im\psi_n}, \tag{29}$$

where δ_n^0 is the nth ring-planet mesh deflection without considering the elastic deformation of the ring gear, as given by (15). According to (29), $V_{-m}^1 = \bar{V}_m^1$.

Rotational Modes When the unperturbed eigenfunction $\mathbf{q}^0$ from (26) is a discrete rotational mode, the translational motions of the sun, carrier and ring are zero and all the planets have the same deflections [5]

$$\mathbf{q}^0 = \left[0,0,u_r^0,0,0,u_c^0,0,0,u_s^0,\xi_1^0,\eta_1^0,u_1^0,\cdots,\xi_1^0,\eta_1^0,u_1^0\right]^T. \tag{30}$$

In the absence of any rigid constraints on any degrees of freedom (e.g., fixed carrier rotation), six such modes exist, each having a distinct natural frequency. Application of these properties to (29) yields

$$\begin{gathered} V_m^1 = -\frac{\cos\alpha_r - im\sin\alpha_r}{2\pi m^2(m^2-1)^2}\delta_{r1}^0 \sum_{n=1}^{N} e^{-im\psi_n} \\ \delta_{r1}^0 = u_r^0 + \xi_1^0\sin\alpha_r - \eta_1^0\cos\alpha_r - u_1^0 \end{gathered} \tag{31}$$

Because the planets are equally spaced with $\psi_n = 2\pi(n-1)/N$, the identity $\sum_{n=1}^{N} e^{-im\psi_n} = 0$ holds for $m \neq jN$, where j is an arbitrary nonzero integer. Thus, for $\mathbf{q}^0$ being a discrete rotational mode, the elastic deformation of the ring in the perturbed system contains only the jN nodal diameter components

$$V_m^1 = -\frac{\cos\alpha_r - im\sin\alpha_r}{2\pi m^2(m^2-1)^2} N\delta_{r1}^0, \quad m = \pm N,\cdots,\pm JN. \tag{32}$$

The eigenvalue perturbation μ is determined by the solvability condition of the second of (27) as (with $<\mathbf{Mq}^0,\mathbf{q}^0>=1$)

$$\begin{gathered} \mu = < L_4 v^1, \mathbf{q}^0 > = -\frac{N^2(\delta_{r1}^0)^2}{\pi} \sum_{m=jN}^{j=1,\cdots,J} \gamma_m \\ \gamma_m = \frac{\cos^2\alpha_r + m^2\sin^2\alpha_r}{m^2(m^2-1)^2} \end{gathered} \tag{33}$$

A candidate solution of the second of (27) is proposed as

$$\mathbf{q}^1 = \left[0,0,u_r^1,0,0,u_c^1 0,0,u_s^1,\xi_1^1,\eta_1^1,u_1^1,\cdots,\xi_1^1,\eta_1^1,u_1^1\right]^T. \tag{34}$$

Note that $\mathbf{q}^1$ has the same form as $\mathbf{q}^0$. Use of (34) and the known discrete rotational mode properties reduce (27) to

$$\begin{aligned} &(2\pi k_{rus}/\cos^2\alpha_r + N - \omega_0^2 I_r)u_r^1 + N\delta_{r1}^1 = \\ &\mu I_r u_r^0 - \tfrac{N}{\pi}\delta_{r1}^0 \sum_{m=jN}^{j=1,\cdots,J} \gamma_m \end{aligned}, \tag{35}$$

$$(k_{cu} + Nk_p - \omega_0^2 I_c)u_c^1 - Nk_p\eta_1^1 = \mu I_c u_c^0, \tag{36}$$

$$\begin{aligned} &(k_{su} + Nk_{sp} - \omega_0^2 I_s)u_s^1 - Nk_{sp}(u_1^1 - \xi_1^1\sin\alpha_s \\ &+ \eta_1^1\cos\alpha_s) = \mu I_s u_s^0 \end{aligned}, \tag{37}$$

$$\begin{aligned} &N\left[(\mathbf{K}_{c2}^1)^{\mathrm{T}}\mathbf{p}_c^1 + (\mathbf{K}_{r2}^1)^{\mathrm{T}}\mathbf{p}_r^1 + (\mathbf{K}_{s2}^1)^{\mathrm{T}}\mathbf{p}_s^1 + (\mathbf{K}_{pp} - \omega_0^2\mathbf{M}_p)\mathbf{p}_1^1\right] \\ &= \mu N\mathbf{M}_p\mathbf{p}_1^0 - \mathbf{b}_p\tfrac{N}{\pi}\delta_{r1}^0 \sum_{m=jN}^{j=1,\cdots,J} \gamma_m \end{aligned}. \tag{38}$$

Expressing (35)-(38) in matrix form yields the 6×6 linear system

$$\mathbf{A}_{rot}\mathbf{p}_{rot}^1 = \mathbf{b}_{rot}, \tag{39}$$

$$\mathbf{p}_{rot}^1 = \left[u_r^1,u_c^1,u_s^1,\xi_1^1,\eta_1^1,u_1^1\right]^T, \tag{40}$$

$$\mathbf{b}_{rot} = \mu \mathbf{M}_{rot} \mathbf{p}^0_{rot} - \frac{N}{\pi} \delta^0_{r1} \sum_{m=jN}^{j=1,\cdots,J} \gamma_m \left[1,0,0,\mathbf{b}_p^T\right]^T, \qquad \mathbf{M}_{rot} = diag(I_r, I_c, I_s, N\mathbf{M}_p) \tag{41}$$

One can show that the solvability condition of (39) is identical to (33), so it is satisfied. This guarantees that the solution of the second of (27) has the assumed form (34). The normalization condition $< \mathbf{q}^1, \mathbf{M}\mathbf{q}^0 >= 0$ becomes $< \mathbf{p}^1_{rot}, \mathbf{M}_{rot}\mathbf{p}^0_{rot} >= 0$ in this problem. This and (39) yield

$$\begin{bmatrix} \mathbf{A}_{rot} \\ (\mathbf{M}_{rot}\mathbf{p}^0_{rot})^T \end{bmatrix} \mathbf{p}^1_{rot} = \begin{bmatrix} \mathbf{b}_{rot} \\ 0 \end{bmatrix} \quad \Rightarrow \hat{\mathbf{A}}_{rot}\mathbf{p}^1_{rot} = \hat{\mathbf{b}}_{rot}. \tag{42}$$

The solution of (42) is $\mathbf{p}^1_{rot} = (\hat{\mathbf{A}}^T_{rot}\hat{\mathbf{A}}_{rot})^{-1}\hat{\mathbf{A}}^T_{rot}\hat{\mathbf{b}}_{rot}$. This completes the solution for $\mathbf{q}^1$ in (27).

Collecting results, we have six eigenfunctions $\mathbf{a}$ in (24) with the form

$$\mathbf{a} = \begin{bmatrix} \varepsilon \sum_{m=jN}^{j=\pm 1,\cdots,\pm J} V^1_m e^{im\theta} \\ \mathbf{q}^0 + \varepsilon \mathbf{q}^1 \end{bmatrix}. \tag{43}$$

The discrete elements of the planetary gear (including the ring rigid body motion) deflect as in the discrete rotational modes described in ....[5]. The elastic ring deflection contains only the jN nodal diameter components. The natural frequencies of these modes are distinct.

Translational Modes When the unperturbed eigenfunction from (26) is a discrete translational mode, the eigenvalues are repeated with multiplicity two and the rotational motions of the carrier, sun and ring are zero [5]. The pair of degenerate vibration modes $\widehat{\mathbf{q}}^0$ and $\hat{\mathbf{q}}^0$ satisfy

$$\widehat{\mathbf{q}}^0 = \left[\widehat{\mathbf{p}}^0_r, \widehat{\mathbf{p}}^0_c, \widehat{\mathbf{p}}^0_s, \widehat{\mathbf{p}}^0_1, \cdots, \widehat{\mathbf{p}}^0_N\right]^T, \quad \hat{\mathbf{q}}^0 = \left[\hat{\mathbf{p}}^0_r, \hat{\mathbf{p}}^0_c, \hat{\mathbf{p}}^0_s, \hat{\mathbf{p}}^0_1, \cdots, \hat{\mathbf{p}}^0_N\right]^T, \tag{44}$$

$$\widehat{\mathbf{p}}^0_j = \left[x^0_j, y^0_j, 0\right]^T, \quad \hat{\mathbf{p}}^0_j = \left[y^0_j, -x^0_j, 0\right]^T, \quad j = r, c, s. \tag{45}$$

When the planets are located at $\psi_n = 2\pi(n-1)/N$, the nth planet displacements $\widehat{\mathbf{p}}^0_n$, $\hat{\mathbf{p}}^0_n$ are related as

$$\begin{bmatrix} \widehat{\mathbf{p}}^0_n \\ \hat{\mathbf{p}}^0_n \end{bmatrix} = \begin{bmatrix} \cos\psi_n \mathbf{I} & \sin\psi_n \mathbf{I} \\ -\sin\psi_n \mathbf{I} & \cos\psi_n \mathbf{I} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{p}}^0_1 \\ \hat{\mathbf{p}}^0_1 \end{bmatrix}, \quad n = 1, 2, \cdots, N, \tag{46}$$

where $\mathbf{I}$ is a 3×3 identity matrix. Six such eigensolution pairs exist.

The degenerate unperturbed eigenvalue ω_0^2 of multiplicity two in (24) has two orthonormal, unperturbed eigenfunctions $\widehat{\mathbf{a}}^0$ and $\hat{\mathbf{a}}^0$ of the extended operator form (10). As a consequence, the unperturbed eigenfunction $\mathbf{a}^0$ is a linear combination of $\widehat{\mathbf{a}}^0$ and $\hat{\mathbf{a}}^0$

$$\mathbf{a}^0 = c_1 \widehat{\mathbf{a}}^0 + c_2 \hat{\mathbf{a}}^0, \quad \widehat{\mathbf{a}}^0 = \begin{bmatrix} 0 \\ \widehat{\mathbf{q}}^0 \end{bmatrix}, \quad \hat{\mathbf{a}}^0 = \begin{bmatrix} 0 \\ \hat{\mathbf{q}}^0 \end{bmatrix} \tag{47}$$

where c_1 and c_2 are constants. Analogous to the procedure for the rotational mode, use of the discrete translational mode properties reduces (29) to

$$V^1_m = -\frac{\cos\alpha_r - im\sin\alpha_r}{4\pi m^2(m^2-1)^2} \left[\upsilon \sum_{n=1}^{N} e^{-i(m-1)\psi_n} + \bar{\upsilon} \sum_{n=1}^{N} e^{-i(m+1)\psi_n} \right], \qquad \upsilon = (c_1 + ic_2)(A_1 - iA_2)$$

$$\begin{aligned} A_1 &= y^0_r \cos\alpha_r - x^0_r \sin\alpha_r + \widehat{\xi}^0_1 \sin\alpha_r - \widehat{\eta}^0_1 \cos\alpha_r - \widehat{u}^0_1 \\ A_2 &= -y^0_r \sin\alpha_r - x^0_r \cos\alpha_r + \hat{\xi}^0_1 \sin\alpha_r - \hat{\eta}^0_1 \cos\alpha_r - \hat{u}^0_1, \end{aligned} \tag{48}$$

where $\bar{\upsilon}$ is the complex conjugate of υ. $\sum_{n=1}^{N} e^{-i(m-1)\psi_n}$ being zero requires $m \neq jN+1$, where j is an arbitrary integer; $\sum_{n=1}^{N} e^{-i(m+1)\psi_n}$being zero requires $m \neq jN-1$. Thus, V^1_m vanishes if and only if $m \neq jN \pm 1$. This yields the following rule: The elastic deformation of the ring for elastic translational modes contains only $jN \pm 1$ nodal diameter components.

The solvability conditions of the second equation of (27) form a 2×2 algebraic eigenvalue problem $\mathbf{D}_r\mathbf{c} = \mu\mathbf{c}$, where $\mathbf{c} = (c_1, c_2)^T$. $\mathbf{D}_r$ is diagonal with the repeated eigenvalues

$$\mu_1 = \mu_2 = -\frac{N^2(A_1^2 + A_2^2)}{4\pi} \sum_{m=jN+1}^{\gamma_m}, \tag{49}$$

where here (and in all subsequent summations) j is an integer such that m takes only values within the range specified in (22), i.e., $-JN \leq m \leq JN$ and $m \neq -1, 0, 1$. Thus, the eigenvalues for the elastic ring model remain degenerate and c_1, c_2 are indeterminate.

The eigenfunction perturbation is proposed as $\mathbf{q}^1 = c_1\widehat{\mathbf{q}}^1 + c_2\hat{\mathbf{q}}^1$, where $\widehat{\mathbf{q}}^1$ and $\hat{\mathbf{q}}^1$ are a pair of vectors having the same properties (44)-(46) as the discrete translational modes. Substitution of $\mathbf{q}^1$ into (27) yields a set of simplified equations that, if satisfied, ensures (27) is satisfied for any c_1 and c_2. The perturbation equations from (27) for the sun, carrier and ring rigid motion reduce to the six equations (50)-(55), and the perturbation equations for all the planets reduce to (56) and (57)

$$(\pi k_{rbs} + \pi k_{rus} + \tfrac{N}{2} - \omega_0^2 m_r) x_r^1 - \tfrac{N}{2}(\widehat{\sigma}_r^1 \sin\alpha_r + \hat{\sigma}_r^1 \cos\alpha_r) = \mu m_r x_r^0 + \beta_1 \tag{50}$$

$$(\pi k_{rbs} + \pi k_{rus} + \tfrac{N}{2} - \omega_0^2 m_r) y_r^1 + \tfrac{N}{2}(\widehat{\sigma}_r^1 \cos\alpha_r - \hat{\sigma}_r^1 \sin\alpha_r) = \mu m_r y_r^0 + \beta_2 \tag{51}$$

$$(k_c + N k_{pn} - \omega_0^2 m_c) x_c^1 + \frac{N}{2} k_p (-\widehat{\xi}_1^1 + \hat{\eta}_1^1) = \mu m_c x_c^0, \tag{52}$$

$$(k_c + N k_{pn} - \omega_0^2 m_c) y_c^1 + \frac{N}{2} k_p (-\hat{\xi}_1^1 - \widehat{\eta}_1^1) = \mu m_c y_c^0, \tag{53}$$

$$(k_s + \tfrac{N}{2} k_{sp} - \omega_0^2 m_s) x_s^1 + \tfrac{N}{2} k_{sp} (-\widehat{\sigma}_s^1 \sin\alpha_s + \hat{\sigma}_s^1 \cos\alpha_s) = \mu m_s x_s^0, \tag{54}$$

$$(k_s + \tfrac{N}{2} k_{sp} - \omega_0^2 m_s) y_s^1 + \tfrac{N}{2} k_{sp} (-\widehat{\sigma}_s^1 \cos\alpha_s - \hat{\sigma}_s^1 \sin\alpha_s) = \mu m_s y_s^0, \tag{55}$$

$$k_{pn} \widehat{\mathbf{p}}_c^1 + \mathbf{K}_{r4}^1 \widehat{\mathbf{p}}_r^1 + \mathbf{K}_{s4}^1 \widehat{\mathbf{p}}_s^1 + (\mathbf{K}_{pp} - \omega_0^2 \mathbf{M}_p) \widehat{\mathbf{p}}_1^1 = \mu \mathbf{M}_p \widehat{\mathbf{p}}_1^0 + \beta_3 \mathbf{b}_p, \tag{56}$$

$$k_{pn} \hat{\mathbf{p}}_c^1 + \mathbf{K}_{r4}^1 \hat{\mathbf{p}}_r^1 + \mathbf{K}_{s4}^1 \hat{\mathbf{p}}_s^1 + (\mathbf{K}_{pp} - \omega_0^2 \mathbf{M}_p) \hat{\mathbf{p}}_1^1 = \mu \mathbf{M}_p \hat{\mathbf{p}}_1^0 + \beta_4 \mathbf{b}_p, \tag{57}$$

where

$$\begin{aligned} \widehat{\sigma}_r^1 &= \widehat{\xi}_1^1 \sin\alpha_r - \widehat{\eta}_1^1 \cos\alpha_r - \widehat{u}_1^1 \\ \hat{\sigma}_r^1 &= \hat{\xi}_1^1 \sin\alpha_r - \hat{\eta}_1^1 \cos\alpha_r - \hat{u}_1^1 \end{aligned} \tag{58}$$

$$\begin{aligned} \widehat{\sigma}_s^1 &= -\widehat{\xi}_1^1 \sin\alpha_s - \widehat{\eta}_1^1 \cos\alpha_s + \widehat{u}_1^1 \\ \hat{\sigma}_s^1 &= -\hat{\xi}_1^1 \sin\alpha_s - \hat{\eta}_1^1 \cos\alpha_s + \hat{u}_1^1 \end{aligned} \tag{59}$$

$$\begin{aligned} \beta_1 &= \frac{-N^2(A_1 \sin\alpha_r + A_2 \cos\alpha_r)}{4\pi} \sum_{m=jN+1} \gamma_m, \\ \beta_2 &= \frac{N^2(A_1 \cos\alpha_r - A_2 \sin\alpha_r)}{4\pi} \sum_{m=jN+1} \gamma_m \end{aligned} \tag{60}$$

$$\beta_3 = \frac{N A_1}{2\pi} \sum_{m=jN+1} \gamma_m, \quad \beta_4 = \frac{N A_2}{2\pi} \sum_{m=jN+1} \gamma_m, \tag{61}$$

$$\mathbf{K}_{r4}^1 = \begin{bmatrix} -\sin^2\alpha_r & \sin\alpha_r \cos\alpha_r & 0 \\ \sin\alpha_r \cos\alpha_r & \cos^2\alpha_r & 0 \\ \sin\alpha_r & \cos\alpha_r & 0 \end{bmatrix},$$

$$\mathbf{K}_{s4}^1 = k_{sp} \begin{bmatrix} -\sin^2\alpha_s & -\sin\alpha_s \cos\alpha_s & 0 \\ -\sin\alpha_s \cos\alpha_s & -\cos^2\alpha_s & 0 \\ \sin\alpha_s & \cos\alpha_s & 0 \end{bmatrix}$$

$\widehat{\sigma}_r^1$ and $\widehat{\sigma}_s^1$ are the deflections of the first planet in the direction of the lines of action for the ring-planet and sun-planet meshes, respectively. The superscript 1 denotes the first order perturbation.

Expressing (50)-(57) in matrix form after multiplying (56) and (57) by $N/2$, the second equation of (27) reduces to the 12×12 linear system

$$\mathbf{A}_{trn} \mathbf{p}_{trn}^1 = \mathbf{b}_{trn}, \tag{62}$$

$$\mathbf{p}_{trn}^1 = \left[x_r^1, y_r^1, x_c^1, y_c^1, x_s^1, y_s^1, \widehat{\xi}_1^1, \widehat{\eta}_1^1, \widehat{u}_1^1, \hat{\xi}_1^1, \hat{\eta}_1^1, \hat{u}_1^1 \right]^T, \tag{63}$$

$$\mathbf{b}_{trn} = \mu \mathbf{M}_{trn} \mathbf{p}_{trn}^0 + \left[\beta_{t1}, \beta_{t2}, 0, 0, 0, 0, \beta_{t3} \mathbf{b}_p^T, \beta_{t4} \mathbf{b}_p^T \right]^T, \tag{64}$$

$$\mathbf{M}_{trn} = diag(1, 1, m_c, m_c, m_s, m_s, \frac{N}{2} \mathbf{M}_p, \frac{N}{2} \mathbf{M}_p). \tag{65}$$

One can show that the solvability condition of (62) is identical to (49), so it is satisfied. Thus, (27) is satisfied for the given $\mathbf{q}^1$ independent of c_1 and c_2 (which remain indeterminate), and the perturbation $\mathbf{q}^1$ has the same form as $\mathbf{q}^0$.

In summary, there are six degenerate pairs of eigenfunctions $\mathbf{a}$ in (24) with the form

$$\mathbf{a} = \begin{bmatrix} \varepsilon \sum_{m=jN+1} V_m^1 e^{im\theta} + c.c. \\ \mathbf{q}^0 + \varepsilon \mathbf{q}^1 \end{bmatrix} = \begin{bmatrix} \varepsilon \sum_{m=jN\pm1} V_m^1 e^{im\theta} \\ \mathbf{q}^0 + \varepsilon \mathbf{q}^1 \end{bmatrix}. \tag{66}$$

Note that terms associated with $m = jN-1$ in (66) are the complex conjugate terms for $m = jN+1$. The discrete elements of the planetary gear (including the ring rigid body motion) deflect as in the translational modes described in ....[5]. The elastic ring deflection contains only the $jN \pm 1$ nodal diameter components. The natural frequencies of these modes are degenerate.

Planet Modes For $N \geq 4$, the unperturbed system has three unperturbed eigenvalues associated with the discrete planet modes, and each of them is degenerate with multiplicity$N-3$. For these modes, the sun, carrier and rigid ring motions are zero. The deflections of the planets are proportional with $\mathbf{p}_n^0 = w_n^l \mathbf{p}_1^0$, where the $N-3$ sets of coefficients satisfy [5]

$$\sum_{n=1}^{N} w_n^l = 0, \quad \sum_{n=1}^{N} w_n^l \cos\psi_n = 0, \quad \sum_{n=1}^{N} w_n^l \sin\psi_n = 0, \quad l = 1,\cdots,N-3. \tag{67}$$

When N is odd, the $N-3$ solutions of (67) are

$$w_n^{2s-3} = \cos s\psi_n, \quad w_n^{2s-2} = \sin s\psi_n, \quad s = 2,\cdots,\frac{N-1}{2}. \tag{68}$$

When N is even, the $N-3$ solutions of (67) consist of (68) for $s = 2,\cdots,int(\frac{N-1}{2})$ and the additional solution

$$w_n^{N-3} = \cos\frac{N}{2}\psi_n. \tag{69}$$

A general discrete planet mode of the unperturbed system is the linear combination $\mathbf{q}^0 = \sum_{l=1}^{N-3} d_l \mathbf{q}_l^0$, with

$$\mathbf{q}_l^0 = \left[\mathbf{0},\mathbf{0},\mathbf{0},w_1^l\mathbf{p}_1^0,\cdots,w_N^l\mathbf{p}_1^0\right]^T \tag{70}$$

With this mode, reduction of (29) yields the elastic deformation of the ring as

$$V_m^1 = -\sigma_r^0 \frac{\cos\alpha_r - im\sin\alpha_r}{2\pi m^2(m^2-1)^2} \sum_{l=1}^{N-3} (d_l \sum_{n=1}^{N} w_n^l e^{-im\psi_n}), \quad \sigma_r^0 = \xi_1^0 \sin\alpha_r - \eta_1^0\cos\alpha_r - u_1^0 \tag{71}$$

The $N-3$ solvability conditions for the second of (27) give

$$\mathbf{D}_p\mathbf{d} = \mu\mathbf{d} \quad , \quad \mathbf{d}=[d_1,d_2,\cdots,d_{N-3}]^T, \tag{72}$$

$$\mathbf{D}_p = [D_{tj}]_{(N-3)\times(N-3)} = -\frac{(\sigma_r^0)^2}{2\pi} \left[\sum_{m=\pm 2}^{\pm JN} \gamma_m (\sum_{n=1}^{N} w_n^t e^{-im\psi_n})(\sum_{n=1}^{N} w_n^j e^{im\psi_n})\right]_{(N-3)\times(N-3)} . \tag{73}$$

Although the elements D_{tj} of $\mathbf{D}_p$ appear complicated, use of the solutions (68) and (69) simplifies them. $\mathbf{D}_p$ is diagonal, yielding closed-form expressions for μ. When N is odd, the first-order eigenvalue perturbations are

$$\mu_{2s-3} = \mu_{2s-2} = -\frac{N^2(\sigma_r^0)^2}{4\pi} \sum_{m=jN+s} \gamma_m, \quad s = 2,\cdots,\frac{N-1}{2}, \tag{74}$$

When N is even, (74) holds for $s = 2,\cdots,\frac{N}{2}-1$, and the remaining eigenvalue perturbation is

$$\mu_{N-3} = -\frac{N^2(\sigma_r^0)^2}{4\pi} \sum_{m=jN+\frac{N}{2}} \gamma_m. \tag{75}$$

For each of the three unperturbed discrete planet modes with multiplicity $N-3$, the corresponding perturbed eigenfunctions evolve into $int(\frac{N-3}{2})$ pairs of degenerate planet modes for arbitrary N and one additional distinct planet mode for even N.

For degenerate planet modes with natural frequency perturbation from (74), the unperturbed eigenfunction is a linear combination of two instead of $N-3$ modes in (70). According to this and (68), (71) reduces to

$$V_m^1 = -\sigma_r^0 \tfrac{\cos\alpha_r - im\sin\alpha_r}{2\pi m^2(m^2-1)^2} \sum_{n=1}^{N} \left(d_{2s-3}\cos s\psi_n e^{-im\psi_n} + d_{2s-2}\sin s\psi_n e^{-im\psi_n}\right) \tag{76}$$

V_m^1 is zero when $m \neq jN \pm s$. This yields a rule governing the nodal diameter components of the ring modal deflections for a mode with given $s \in \{2,\cdots,int(\frac{N-1}{2})\}$: the elastic deformation of the ring for degenerate planet modes contains only $jN \pm s$ nodal diameter components. The nonzero nodal diameter components are

$$V_m^1 = -N\sigma_r^0 \frac{\cos\alpha_r - im\sin\alpha_r}{4\pi m^2(m^2-1)^2}(d_{2s-3} - id_{2s-2}), \quad m = jN+s, \tag{77}$$

$$V_m^1 = -N\sigma_r^0 \frac{\cos\alpha_r - im\sin\alpha_r}{4\pi m^2(m^2-1)^2}(d_{2s-3} + id_{2s-2}), \quad m = jN-s. \tag{78}$$

For distinct planet modes whose natural frequency is $\omega^2 = \omega_0^2 + \varepsilon\mu_{\frac{N}{2}}$ (exist only for even N), (71) reduces to

$$V_m^1 = -\sigma_r^0 \tfrac{\cos\alpha_r - im\sin\alpha_r}{2\pi m^2(m^2-1)^2} \sum_{n=1}^{N} \cos(\tfrac{N}{2}\psi_n)e^{-im\psi_n} = -N\sigma_r^0 \tfrac{\cos\alpha_r - im\sin\alpha_r}{4\pi m^2(m^2-1)^2} \quad \text{for all } m = jN \pm \tfrac{N}{2}. \tag{79}$$

Terms in the first expression for V_m^1 in (79) vanish for $m \neq jN \pm \frac{N}{2}$. Accordingly, the perturbed eigenfunction contains only $jN \pm \frac{N}{2}$ nodal diameter components ($s = \frac{N}{2}$).

For the degenerate eigensolution $\mu_{2s-3} = \mu_{2s-2}$ with specified $s \in \{2,3,\cdots,int(\frac{N-1}{2})\}$, the eigenfunction perturbation $\mathbf{q}^1$ is proposed as the linear combination

$$\mathbf{q}^1 = d_{2s-3}\widehat{\mathbf{q}}^1 + d_{2s-2}\hat{\mathbf{q}}^1, \tag{80}$$

$$\begin{aligned} \widehat{\mathbf{q}}^1 &= \left[\mathbf{0},\mathbf{0},\mathbf{0},z_1^1(\mathbf{p}_1^1)^T,\cdots,z_N^1(\mathbf{p}_1^1)^T\right]^T, \\ \hat{\mathbf{q}}^1 &= \left[\mathbf{0},\mathbf{0},\mathbf{0},z_1^2(\mathbf{p}_1^1)^T,\cdots,z_N^2(\mathbf{p}_1^1)^T\right]^T \end{aligned} \tag{81}$$

where $\widehat{\mathbf{q}}^1$ and $\hat{\mathbf{q}}^1$ have the same form as the discrete planet modes in (70). Substituting this form of $\mathbf{q}^1$ into (27), the equations associated with the sun, carrier and ring rigid motion lead to three equations identical to (67) except $w_n^l \rightarrow z_n^l$, but here $l = 1,2$. The solutions for z_n^1, z_n^2 are

$$z_n^1 = \cos s\psi_n \quad , \quad z_n^2 = \sin s\psi_n, \quad n = 1,2,\cdots,N. \tag{82}$$

The remaining equations of (27) (the ones associated with deflections of the planets) yield

$$(\mathbf{K}_{pp} - \omega_0^2\mathbf{M}_p)\mathbf{p}_1^1 = \mu_{2s-3}\mathbf{M}_p\mathbf{p}_1^0 + \frac{N\sigma_r^0}{2\pi}\mathbf{b}_p \sum_{m=jN+s}\gamma_m. \tag{83}$$

One can show that the solvability condition of (83) is identical to (74) and (75). Thus, $\mathbf{p}_1^1$ is solved from (83), which, with (82), completes the solution for $\mathbf{q}^1$. This ensures $\mathbf{q}^1$ has the structure of a discrete planet mode.

For the distinct eigenvalue μ_{N-3} in (75), one can similarly show that the eigenfunction perturbation $\mathbf{q}^1$ has the form of a discrete planet mode.

In summary, for each $s \in \{2,3,\cdots,int(\frac{N-1}{2})\}$ there are three degenerate pairs of eigenfunctions $\mathbf{a}_s$ in (24) with the form

$$\mathbf{a}_s = \begin{bmatrix} \varepsilon \sum\limits_{m=jN+s} V_m^1 e^{im\theta} + c.c. \\ \mathbf{q}^0 + \varepsilon\mathbf{q}^1 \end{bmatrix}. \tag{84}$$

For even N, an additional three distinct eigenfunctions are present with the form of (84) and $s = \frac{N}{2}$. The discrete elements of the planetary gear (including the ring rigid body motion) deflect as in the planet modes described in [5]. The elastic ring deflection contains only the $jN \pm s$ nodal diameter components.

Elastic Ring Perturbation

ERP is the complementary case of DPP. The stiffness of the ring-planet mesh is unity in both cases (from (6)). In ERP, the ring bending stiffness is $O(1)$ while in DPP it is $O(1/\varepsilon)$; the stiffnesses of all the remaining meshes/bearings are $O(1/\varepsilon)$ while in DPP they are $O(1)$. The perturbation parameter is defined by $\varepsilon = 1/k_{sp}$. A perturbation process similar to (24)-(27) yields the perturbation equations for $\mathbf{a}^0$

$$-\omega_0^2(1+\partial^2/\partial\theta^2)v^0/(2\pi) + k_{bend}L_1 v^0 + L_2 v^0 + L_3\mathbf{q}^0 = 0, \tag{85}$$

$$\mathbf{K}_{rb}\mathbf{p}_r^0 = \mathbf{0}, \quad \mathbf{K}_{cb}\mathbf{p}_c^0 + \sum_n \mathbf{K}_{c2}^n\mathbf{p}_n^0 = \mathbf{0}, \tag{86}$$

$$\begin{aligned} (\mathbf{K}_{sb} + \sum_n \mathbf{K}_{s1}^n)\mathbf{p}_s^0 + \sum_n \mathbf{K}_{s2}^n\mathbf{p}_n^0 = \mathbf{0}, \\ (\mathbf{K}_{c2}^n)^T\mathbf{p}_c^0 + (\mathbf{K}_{s2}^n)^T\mathbf{p}_s^0 + \mathbf{K}_{pp}\mathbf{p}_n^0 = \mathbf{0} \end{aligned} \tag{87}$$

Equations (86)-(87) form a problem as $\mathbf{A}\mathbf{q}^0 = \mathbf{0}$. One can prove that $\mathbf{A}$ is positive definite so $\mathbf{q}^0 = \mathbf{0}$. Accordingly, the last item in (85) vanishes, so the unperturbed system is an elastic ring having equally spaced spring supports with elimination of the three rigid body motions as indicated in (4). The unperturbed eigenfunction is

$$\mathbf{a}_0 = \begin{bmatrix} v^0 \\ \mathbf{0} \end{bmatrix}^T. \tag{88}$$

Equations (28) and (88) are the unperturbed eigenfunctions from DPP and ERP, respectively. Together they form a non-overlapping, complete (in the mathematical sense) basis for the linear space of extended variables $\mathbf{a} = \left[v,\mathbf{q}^T\right]^T$. This suggests the set of perturbed eigenfunctions from DPP and ERP forms a complete set of vibration modes for planetary gears having elastic ring gears. This conclusion is made rigorous subsequently.

The perturbation equations for $\mathbf{a}^1$ are

$$\mathbf{K}_{rb}\mathbf{p}_r^1 = -\sum_n \mathbf{b}_r\left(\frac{\partial v^0}{\partial\theta}\sin\alpha_r + v^0\cos\alpha_r\right)\bigg|_{\theta=\psi_n}, \tag{89}$$

$$\mathbf{K}_{cb}\mathbf{p}_c^1 + \sum_n \mathbf{K}_{c2}^n\mathbf{p}_n^1 = \mathbf{0}, \quad (\mathbf{K}_{sb} + \sum_n \mathbf{K}_{s1}^n)\mathbf{p}_s^1 + \sum_n \mathbf{K}_{s2}^n\mathbf{p}_n^1 = \mathbf{0}, \tag{90}$$

$$\begin{aligned} &(\mathbf{K}_{c2}^n)^T\mathbf{p}_c^1 + (\mathbf{K}_{s2}^n)^T\mathbf{p}_s^1 + \mathbf{K}_{pp}\mathbf{p}_n^1 = -\mathbf{b}_p \\ &\left(\tfrac{\partial v^0}{\partial\theta}\sin\alpha_r + v^0\cos\alpha_r\right)\Big|_{\theta=\psi_n} \end{aligned}, \tag{91}$$

$$-\frac{\omega_0^2}{2\pi}(1+\frac{\partial^2}{\partial\theta^2})v^1 + k_{bend}L_1 v^1 + L_2 v^1 = \frac{\mu}{2\pi}(v^0 + \frac{\partial^2 v^0}{\partial\theta^2}) - L_3\mathbf{q}^1 . \quad (92)$$

We draw on the modal properties of a ring on a general elastic foundation as determined analytically in ...[13], where the modal expressions for rings having equally spaced springs are given. In the unperturbed problem (85), each spring is oriented with an angle of $\pi/2-\alpha_r$ to the radial direction. With elimination of the ring rigid body motions, the ring deflection is represented as (22). Thus, $2JN-2$ unperturbed modes exist. For a free ring with no supports, all the natural frequencies are degenerate with multiplicity two. When the ring has equally spaced springs, some natural frequencies split and the others remain degenerate. The unperturbed modes of the ERP are classified into four types based on the nodal diameter components they contain: type 0, type 1, type s and single nodal diameter component modes [13].

For brevity, only *type 0* modes are considered. They are linear combinations of the jN nodal diameter components, $v^0 = \sum_{j=1}^{J} V_{jN}^d \cos jN\theta$. Such a mode exists for each of the J values of $d = N, 2N, \cdots, JN$, where d indicates the dominant nodal diameter component. Substitution of this expression for v^0 into (89) yields

$$\mathbf{K}_{rb}\mathbf{p}_r^1 = \left[0, 0, -N \sum_{m=jN}^{j=1,\cdots,J} (m\sin\alpha_r + \cos\alpha_r)V_m^d\right]^T . \quad (93)$$

Because $\mathbf{K}_{rb}$ is diagonal, the first two elements of $\mathbf{p}_r^1$ corresponding to ring rigid translations are zero, which is the same as for a discrete rotational mode. Similar analysis of (90) and (91) for the sun, carrier and planets reveals that $\mathbf{q}^1$ has the form (30) of a discrete rotational mode. The eigenvalue perturbation μ is obtained from the solvability condition of (92). Following lengthy algebra, the solution v^1 of (92) has the same form as v^0. These results show that the perturbed eigenfunction has the properties of a rotational mode as defined earlier.

Similar processes show that when the unperturbed mode v^0 is of type 1 from [13], the perturbed mode of the elastic-discrete model is a translational mode. When the unperturbed mode v^0 is of type s from [13], the perturbed mode is a planet mode. When the unperturbed mode is a single nodal diameter component mode, the perturbed mode is a purely ring mode.

Thus, every unperturbed ERP mode evolves into one of the four modal categories of the elastic-discrete model. The same is true for DPP. The numbers of modes obtained from each of DPP and ERP are $3N$+9 and $2JN-2$, respectively. The total number of eigenfunctions obtained from the perturbation analyses is $3N+2JN+7$, which equals the number of degrees of freedom for arbitrary J in (22). The modal property classification from perturbation analysis exactly matches the properties of the numerical results in Figure 2, Table 1 and Table 2. Evidently, all modes have been included and categorized from the two perturbations.

CANDIDATE MODE METHOD

The foregoing perturbation analysis derives the modal properties by combining two perturbation problems, each having a different perturbation parameter and unperturbed problem. The method appeals to physical reasoning where the elastic-discrete system modes are seen to evolve from known simpler systems. A plausible argument given above heuristically concludes that this approach captures all modes of the general system. Nevertheless, perturbation is inherently linked to small values of the perturbation parameter, and the use of two separate perturbation problems to conclude that all modes are accounted for is not rigorous mathematically. Guided by the foregoing perturbation results, this section derives the general elastic-discrete system modal properties in a rigorous way that is free from any reliance on a small parameter. This alternate derivation assumes eigensolutions having the properties of the four mode types from perturbation and then confirms such eigensolutions satisfy the eignevalue problem. An accounting at the end ensures this approach captures all possible vibration modes.

A candidate rotational mode has the ring deflection

$$v_{rot} = \sum_{j=1}^{J} V_{jN}\cos jN\theta, \quad (94)$$

and discrete element deflection $\mathbf{q}_{rot}$ having the form (30). Substituting (94) into (17), multiplying by $\cos l\theta$, and integrating from 0 to 2π yields

$$-\frac{1+l^2}{2}\omega^2 V_l + \frac{c_l}{2}V_l + N\cos^2\alpha_r \sum_{j=1}^{J} V_{jN} + N\sigma_r\cos\alpha_r = 0, \quad l = N, 2N, \cdots, JN \quad (95)$$

$$c_l = 2\pi k_{bend} l^2(l^2-1)^2 + 2\pi k_{rus} + 2\pi l^2 k_{rbs}, \quad \sigma_r = \xi_1\sin\alpha_r - \eta_1\cos\alpha_r - u_1 \quad (96)$$

Use of the assumed modal properties to reduce (18) yields only one equation for the ring rigid motion

$$(2\pi k_{rus}/\cos^2\alpha_r + N - \omega^2/\cos^2\alpha_r)u_r + N\sigma_r + \cos\alpha_r \sum_{j=1}^{J} V_{jN} = 0 . \quad (97)$$

The remaining equations in (18) vanish. Similarly, (19) and (20) reduce to

$$(k_{cu}+Nk_p-\omega^2 I_c)u_c-Nk_p\eta_1=0, \tag{98}$$

$$\begin{aligned}&(k_{su}+Nk_{sp}-\omega^2 I_s)u_s+Nk_{sp}(-\xi_1\sin\alpha_s-\\&\eta_1\cos\alpha_s+u_1)=0\end{aligned}. \tag{99}$$

With the assumed modal form and algebraic manipulation, (21) becomes

$$\begin{aligned}&(\mathbf{K}_{c2}^1)^T\mathbf{p}_c+(\mathbf{K}_{r2}^1)^T\mathbf{p}_r+(\mathbf{K}_{s2}^1)^T\mathbf{p}_s+\\&(\mathbf{K}_{pp}-\omega^2\mathbf{M}_p)\mathbf{p}_1+\mathbf{b}_p\cos\alpha_r\sum_{j=1}^{J}6V_{jN}=\mathbf{0}\end{aligned}. \tag{100}$$

Equations (95)-(100) form a reduced eigenvalue problem of order $J+6$ with the eigenvector $(V_N,\cdots,V_{JN,}u_r,u_s,u_c,\xi_1,\eta_1,u_1)^T$. In general, the eigenvalues are all distinct (except for specially chosen parameters). From the eigenvectors of the reduced eigenvalue problem, $J+6$ rotational modes of the full system are constructed from (94) and $\mathbf{q}_{rot}$.

A pair of candidate translational modes is

$$\widehat{\mathbf{a}}=\left[\sum_{m=jN+1}V_m e^{im\theta}+c.c.,\widehat{\mathbf{q}}_{trn}^T\right]^T,\quad \hat{\mathbf{a}}=\left[\sum_{m=jN+1}iV_m e^{im\theta}+c.c.,\hat{\mathbf{q}}_{trn}^T\right]^T, \tag{101}$$

where $\widehat{\mathbf{q}}_{trn}$, $\hat{\mathbf{q}}_{trn}$ are a pair of discrete translational modes having the same form as described in (44)-(46). (Recall the note below (49) regarding allowable values of m.) Guided by the perturbation solution in (48), V_m is expressed as $V_m=(\cos\alpha_r-im\sin\alpha_r)U_m$, where U_m is complex.

Substituting $\widehat{\mathbf{a}}$ and $\hat{\mathbf{a}}$ into (17), multiplying by $e^{-il\theta}$, and integrating from 0 to2πyields the equations governing U_m. When $l=jN+1$, there are $2J-1$ equations

$$\begin{aligned}&-(1+l^2)\omega^2U_l+c_lU_l+\frac{N}{2}(A_1-iA_2)+\\&N\sum_{m=jN+1}(\cos^2\alpha_r+m^2\sin^2\alpha_r)U_m=0,\end{aligned} \tag{102}$$

where A_1 and A_2 have the form in (48), and c_l is defined in (96). When $l=jN-1$, the following $2J-1$ equations result

$$\begin{aligned}&-(1+l^2)\omega^2U_l+c_lU_l+\frac{N}{2}(A_1+iA_2)+N\\&\sum_{m=jN-1}(\cos^2\alpha_r+m^2\sin^2\alpha_r)U_m=0.\end{aligned} \tag{103}$$

For other values of l the resulting equations from (17) vanish. For each l in (102), there is a corresponding $-l$ in (103) whose equation is the complex conjugate of (102). Thus, equations (103) and (102) are equivalent. One obtains $4J-2$ real equations because U_l in (102) is complex. Substitution of $\widehat{\mathbf{a}}$ and $\hat{\mathbf{a}}$ into (18)-(21) generates an additional 12 real equations similar to (50)-(57) with the elimination of superscripts 0 or 1, substitution of $\mu=0$, and replacement of β_1, β_2, β_3, β_4 by β_5, β_6, β_7, β_8, respectively,

$$\begin{bmatrix}\beta_5\\\beta_6\end{bmatrix}=\frac{Ne^{-i\alpha_r}}{2}\sum_{m=jN+1}(\cos^2\alpha_r+m^2\sin^2\alpha_r)U_m\cdot\begin{bmatrix}i\\-1\end{bmatrix}+c.c., \tag{104}$$

$$\begin{bmatrix}\beta_7\\\beta_8\end{bmatrix}=\sum_{m=jN+1}(\cos^2\alpha_r+m^2\sin^2\alpha_r)U_m\cdot\begin{bmatrix}1\\i\end{bmatrix}+c.c.. \tag{105}$$

The resulting $4J+10$ real equations form a reduced order eigenvalue problem. Because $\widehat{\mathbf{a}}$ and $\hat{\mathbf{a}}$ are interchangeable, all eigensolutions of the reduced order problem must occur as degenerate eigenvalues with multiplicity two. With these eigensolutions, $2J+5$ pairs of degenerate translational modes are constructed from (101).

A pair of candidate planet modes for a selected $s\in\{2,3,\cdots,int(\frac{N-1}{2})\}$ is

$$\mathbf{a}_{s1}=\left[\sum_{m=jN+s}V_m e^{im\theta}+c.c.,\widehat{\mathbf{q}}_{plt,s}^T\right]^T, \tag{106}$$

$$\mathbf{a}_{s2}=\left[\sum_{m=jN+s}iV_m e^{im\theta}+c.c.,\hat{\mathbf{q}}_{plt,s}^T\right]^T, \tag{107}$$

$$\begin{aligned}\widehat{\mathbf{q}}_{plt,s}^T&=\left[\mathbf{0},\mathbf{0},\mathbf{0},\cos s\psi_1\mathbf{p}_1^T,\cdots,\cos s\psi_N\mathbf{p}_1^T\right]'\\ \hat{\mathbf{q}}_{plt,s}^T&=\left[\mathbf{0},\mathbf{0},\mathbf{0},\sin s\psi_1\mathbf{p}_1^T,\cdots,\sin s\psi_N\mathbf{p}_1^T\right]\end{aligned} \tag{108}$$

where $\widehat{\mathbf{q}}_{plt,s}$, $\hat{\mathbf{q}}_{plt,s}$ are a pair of discrete planet modes having the same form as described in (70). The linear combination $d_{2s-3}\mathbf{a}_{s1}+d_{2s-2}\mathbf{a}_{s2}$ gives the elastic deformation of the ring in the form

$$v=\sum_{m=jN+s}V_m(d_{2s-3}+id_{2s-2})e^{im\theta}+c.c.. \tag{109}$$

Comparing (109) to the perturbation solution (77) and (78) suggests the V_m in (106) and (107) can be written as $V_m=(\cos\alpha_r-$

$im\sin\alpha_r)U_m$ with real U_m and $U_{-m}=U_m$. This is adopted in the candidate modes (106) and (107).

Substituting $\mathbf{a}_{s1}$ into (17), multiplying by $e^{-il\theta}$, and integrating from 0 to2πyields the equations for U_m. When $l=jN+s$, there are $2J$ equations

$$-(1+l^2)\omega^2U_l+c_lU_l+\frac{N}{2}\sigma_r+N\sum_{m=jN+s}(\cos^2\alpha_r +m^2\sin^2\alpha_r)U_m=0, \tag{110}$$

where σ_r is defined in (71). When $l=jN-s$, $2J$ equations result that are equivalent to (110). For other values of l the resulting equations vanish. With the properties of $\mathbf{a}_{s1}$, equations (18)-(20) are satisfied. Substitution of $\mathbf{a}_{s1}$ into (21) yields the same equation for each n

$$(\mathbf{K}_{pp}-\omega^2\mathbf{M}_p)\mathbf{p}_1=-2\sum_{m=jN+s}(\cos^2\alpha_r+m^2\sin^2\alpha_r)U_m. \tag{111}$$

The resulting $2J+3$ equations in (110) and (111) form a reduced order eigenvalue problem with $2J+3$ eigensolutions. Substitution of $\mathbf{a}_{s2}$ into (10) yields the same $2J+3$ order eigenvalue problem. Therefore, each of the $2J+3$ eigensolutions corresponds to a pair of planet modes. Thus, for each s, $2J+3$ pairs of degenerate modes are constructed from (106)-(108). When N is odd there are $\frac{N-3}{2}$ different values of $s\in\{2,3,\cdots,\frac{N-1}{2}\}$, so $(N-3)(2J+3)/2$ degenerate pairs of planet modes are constructed from (106)-(108). When N is even, there are $N/2-2$ different values of $s\in\{2,3,\cdots,\frac{N}{2}-1\}$, so $(N/2-2)(2J+3)$ degenerate pairs of planet modes are similarly constructed.

When N is even, besides the degenerate planet modes, there are additional distinct planet modes. They have the same form as the degenerate planet mode in (106) with $s=N/2$. With some algebraic manipulation of (106), the distinct planet modes have the form

$$\mathbf{a}=\left[\sum_{m=jN+\frac{N}{2}}^{j=0,\cdots,J-1}V_m\cos m\theta,\mathbf{q}^T_{plt,\frac{N}{2}}\right]^T, \quad \mathbf{q}^T_{plt,\frac{N}{2}}=\left[\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{p}_1^T,-\mathbf{p}_1^T,\cdots,\mathbf{p}_1^T,-\mathbf{p}_1^T\right] \tag{112}$$

where V_m is real. A similar reduction as above yields a $J+3$ order eigenvalue problem for the V_m and $\mathbf{p}_1$ with eigenvalue ω^2 from equations (17) and (21). This gives $J+3$ planet modes with distinct eigenvalues from (112). Totally, for even N, there are $(2JN-7J)+(3N-9)$ planet modes constructed from (106)-(108) and (112). Table 2 summarizes the different numbers and types of planet modes.

The final mode type is that of purely ring modes having the (not normalized) form

$$\mathbf{a}=[(\cos\alpha_r\sin m\theta-m\sin\alpha_r\cos m\theta)V_m,\mathbf{0}]^T \quad m=\begin{cases} jN, j=1,\cdots J & \text{for odd or even } N.\\ jN+N/2, j=0,\cdots J-1 & \text{for even } N.\end{cases} \tag{113}$$

In such modes, only the ring gear deforms, and the ring has nodes at all ring-planet mesh locations. All purely ring mode natural frequencies are distinct. Note that a purely ring mode with $m=jN$ in (113) has the same structure as a rotational mode except many elements of the rotational mode are zero. Similarly, a purely ring mode with $m=jN+N/2$ in (113) has the same structure as a distinct planet mode. Rotational modes and purely ring modes with $m=jN$ emerge as split modes of the degenerate eigensolution pairs of a free ring; distinct planet modes and purely ring modes with $m=jN+N/2$ similarly emerge as split modes for even N ...[13].

Substitution of (113) into (17), multiplication by $e^{-il\theta}$, and integration from 0 to 2π yields the following J order diagonal eigenvalue problem for odd N ($2J$ order for even N) with eigenvalue ω^2

$$\left[-(1+l^2)\omega^2+c_l\right]V_l=0, \quad l=\begin{cases} jN, j=1,\cdots J & \text{for odd or even } N,\\ jN+N/2, j=0,\cdots J-1 & \text{for even } N,\end{cases} \tag{114}$$

where c_l is from (96). The remaining equations (18)-(21) vanish for the $\mathbf{a}$ in (113). According to (114), the closed-form natural frequencies expressions are $\omega^2=c_l/(1+l^2)$, where c_l depends on the ring bending stiffness (k_{bend}) and the distributed stiffnesses around the ring circumference (k_{rus} and k_{rbs}). Thus, the natural frequencies of purely ring modes are independent of mesh stiffnesses ($k_{sp}$and k_{rp}). This can also be explained through the gear mesh deflections. The general expressions of sun-planet and ring planet mesh deflections (δ_{sn} and δ_{rn}) are

$$\delta_{sn}=y_s\cos\psi_{sn}-x_s\sin\psi_{sn}-\xi_n\sin\alpha_s-\eta_n\cos\alpha_s +u_s+u_n \tag{115}$$

$$\delta_{rn}=\left.(v\cos\alpha_r+\frac{\partial v}{\partial\theta}\sin\alpha_r)\right|_{\theta=\psi_n}-x_r\sin\psi_{rn}+y_r\cos\psi_{rn}+u_r+ \xi_n\sin\alpha_r-\eta_n\cos\alpha_r-u_n \tag{116}$$

Substitution of (113) into (115) and (116) ensures both the sun-planet and ring-planet mesh deformations are zero.

Overall, four types of modes are identified. For odd N, the numbers of modes for rotational, translational, planet and purely ring modes are $J+6$, $4J+10$, $(2JN-6J)+(3N-9)$, and J, respectively. For even N, they are $J+6$, $4J+10$, $(2JN-7J)+(3N-9)$, and $2J$, respectively. While, the numbers of planet and purely ring modes are different for odd and even numbers of planets, the total number of modes is $(2J+3)N+7$ for either odd or even N. This total equals the total degrees of freedom with $v(\theta,t)$ from (22) and J arbitrarily large. Thus, *all* modes have been categorized.

Furthermore, the numbers of rotational and translational modes are independent of the number of planets N. Changing the number of planets N, while retaining the same J in the ring deformation expansion (22), only changes the numbers of planet modes and purely ring modes. Table 2 lists how the number of planets N affects the number of degenerate and distinct planet modes, and it specifies the planet mode type breakdown for each N. If N increases by one, the total number degrees of freedom increases by $2J+3$ as the total degrees of freedom is $(2J+3)N+7$. If N increases by one from odd to even, $J+3$ of the additional modes are distinct planet modes, and the remaining J additional modes are purely ring modes (the total number of purely ring modes becomes $2J$). If N increases by one from even to odd, all $2J+3$ additional modes are planet modes; furthermore, J purely ring modes change into planet modes. Therefore, the number of planet modes increases by $3J+3$, and the number of purely ring modes decreases by J.

The natural frequency multiplicities and all modal properties from the candidate mode method match the numerical solution in Table 1, Table 2 and Figure 2 (as well as the perturbation results) for arbitrary N and J.

CONCLUSIONS

The distinctive modal properties of planetary gears having equally spaced planets and an elastic continuum ring gear are derived using perturbation analysis and proved using a candidate mode method. The main conclusions are:

1. *All* vibration modes of equally spaced planetary gears having an elastic ring gear are classified into rotational, translational, planet and purely ring modes. For each mode type, the deflections of each planetary gear component, including the elastic ring, are derived in closed-form. In addition, the number of each mode type and multiplicity of the natural frequencies are determined.
2. The modal deflection properties of the sun, carrier and planets for rotational, translational and planet modes are the same as for the discrete model, while the deformation of the ring gear is governed by simple analytical rules dictating which nodal diameter components are present in each mode type.
3. Rotational modes contain jNnodal diameter ring deformation components, while the sun, carrier and ring rigid motion have only rotational motion. All planets have the same displacement. The natural frequencies are distinct.
4. Translational modes contain $jN\pm1$ nodal diameter ring deformation components, while the sun, carrier and ring rigid motion have only translational motion. The deflections of individual planets are related by a rotation matrix.
5. The natural frequencies have multiplicity two.
6. Planet modes contain $jN\pm s$ nodal diameter ring deformation components, where s is one of $2,3,\cdots,int(N/2)$. The translation and rotation of the sun, carrier and rigid ring are zero, and the deflections of the planets are proportional to each other. Most of these natural frequencies have multiplicity two, but some natural frequencies are distinct for an even number of planets.
7. A purely ring mode has only a single nodal diameter ring deformation component. The deflections of all the discrete elements, including the ring rigid motion, are zero. The natural frequencies are distinct.
8. Changing the number of planets N does not affect the number of rotational and translational modes. How the vibration modes are distributed between purely ring modes and planet modes with the addition of a planet depends on whether N changes from odd to even or *vice versa.*

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APPENDIX: NONDIMENSIONAL MATRICES **M** AND **K**

$$\mathbf{M} = diag(\mathbf{M}_r, \mathbf{M}_c, \mathbf{M}_s, \mathbf{M}_1, \cdots, \mathbf{M}_N)$$

$$\mathbf{M}_j = diag(m_j, m_j, I_j/r_j^2), \quad j = c, s, 1, \cdots, N, \quad \mathbf{M}_r = diag[1,\, 1,\, 1/\cos^2\alpha_r]$$

$$\mathbf{K} = \begin{bmatrix} \sum \mathbf{K}_{r1}^n + \mathbf{K}_{rb} & & & \mathbf{K}_{r2}^1 & \cdots & \mathbf{K}_{r2}^N \\ & \sum \mathbf{K}_{c1}^n + \mathbf{K}_{cb} & & \mathbf{K}_{c2}^1 & \cdots & \mathbf{K}_{c2}^N \\ & & \sum \mathbf{K}_{s1}^n + \mathbf{K}_{sb} & \mathbf{K}_{s2}^1 & \cdots & \mathbf{K}_{s2}^N \\ & & & \mathbf{K}_{pp}^1 & & \\ & \text{symmetric} & & & \ddots & \\ & & & & & \mathbf{K}_{pp}^N \end{bmatrix}$$

$$\mathbf{K}_{jb} = diag(k_{jx}, k_{jy}, k_{ju}), \quad j = c, s,$$
$$\mathbf{K}_{rb} = \pi \mathrm{diag}(k_{rbs} + k_{rus}, k_{rbs} + k_{rus}, 2k_{rus}/\cos^2\alpha_r)$$

k_{rbs} and k_{rus} are uniform radial and tangential distributed stiffnesses, respectively. k_{ju}^* is torsional stiffness with units $F-L/rad$ and $k_{ju} = k_{ju}^*/r_j^2$ with units F/L.

$$\mathbf{K}_{pp}^n = \mathbf{K}_{r3}^n + \mathbf{K}_{c3}^n + \mathbf{K}_{s3}^n$$

$$\mathbf{K}_{r1}^n = \begin{bmatrix} \sin^2\psi_{rn} & -\cos\psi_{rn}\sin\psi_{rn} & -\sin\psi_{rn} \\ & \cos^2\psi_{rn} & \cos\psi_{rn} \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\mathbf{K}_{r2}^n = \begin{bmatrix} -\sin\psi_{rn}\sin\alpha_r & \sin\psi_{rn}\cos\alpha_r & \sin\psi_{rn} \\ \cos\psi_{rn}\sin\alpha_r & -\cos\psi_{rn}\cos\alpha_r & -\cos\psi_{rn} \\ \sin\alpha_r & -\cos\alpha_r & -1 \end{bmatrix}$$

$$\mathbf{K}_{r3}^n = \begin{bmatrix} \sin^2\alpha_r & -\cos\alpha_r\sin\alpha_r & -\sin\alpha_r \\ & \cos^2\alpha_r & \cos\alpha_r \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\mathbf{K}_{c1}^n = k_{pn}\begin{bmatrix} 1 & 0 & -\sin\psi_n \\ & 1 & \cos\psi_n \\ \text{symmetric} & & 1 \end{bmatrix},$$
$$\mathbf{K}_{c2}^n = k_{pn}\begin{bmatrix} -\cos\psi_n & \sin\psi_n & 0 \\ -\sin\psi_n & -\cos\psi_n & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{K}_{c3} = diag(k_{pn}, k_{pn}, 0)$$

$$\mathbf{K}_{s1}^n = k_{sp}\begin{bmatrix} \sin^2\psi_{sn} & -\cos\psi_{sn}\sin\psi_{sn} & -\sin\psi_{sn} \\ & \cos^2\psi_{sn} & \cos\psi_{sn} \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\mathbf{K}_{s2}^n = k_{sp}\begin{bmatrix} \sin\psi_{sn}\sin\alpha_s & \sin\psi_{sn}\cos\alpha_s & -\sin\psi_{sn} \\ -\cos\psi_{sn}\sin\alpha_s & -\cos\psi_{sn}\cos\alpha_s & \cos\psi_{sn} \\ -\sin\alpha_s & -\cos\alpha_s & 1 \end{bmatrix}$$

$$\mathbf{K}_{s3}^n = k_{sp}\begin{bmatrix} \sin^2\alpha_s & \cos\alpha_s\sin\alpha_s & -\sin\alpha_s \\ & \cos^2\alpha_s & -\cos\alpha_s \\ \text{symmetric} & & 1 \end{bmatrix}$$

$$\psi_{sn} = \psi_n - \alpha_s, \quad \psi_{rn} = \psi_n + \alpha_r$$

Table 1. Dimensional parameters and dimensionless natural frequencies of a planetary gear with six equally spaced planets. The designations R, T, P and PR denote rotational, translational, planet and purely ring modes.

Inertias (kg)	$I_r/r_r^2 = 8.891$, $I_c/r_c^2 = 6.000$, $I_s/r_s^2 = 2.500$, $I_p/r_p^2 = 2.000$
Masses (kg)	$m_r = 7.350$, $m_c = 5.430$, $m_s = 0.400$, $m_p = 1.000$
Stiffnesses (N/m)	$k_{rp} = k_{sp} = 10^8$, $k_{rbs} = k_{rus} = 0$, $k_{bend} = 5 \times 10^6$, $k_s = k_{su} = 5 \times 10^7$, $k_c = k_{cu} = 5 \times 10^{11}$, $k_p = 10^9$
Pressure angle (deg)	$\alpha_r = \alpha_s = 24.60$
Dimensionless natural frequencies	$\omega_1 = 0.1520$(R), $\omega_{2,3} = 0.1871$(T), $\omega_{4,5} = 0.6472$(P), $\omega_6 = 1.0227$(P), $\omega_{7,8} = 1.0231$(T), $\omega_9 = 1.1009$(R), $\omega_{10,11} = 1.1695$(P), $\omega_{12} = 1.6971$(PR), $\omega_{13} = 1.8549$(P), $\omega_{14} = 1.9161$(R)

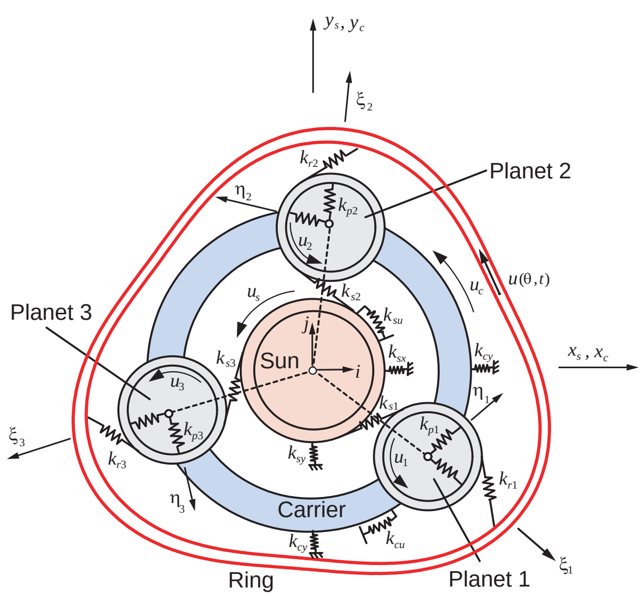

Figure 1. Elastic-discrete model of a planetary gear and corresponding system coordinates. The distributed springs around the ring circumference are not shown.

Table 2. Number of planet modes in different sub-types for different number of planets N , where $\times$ denotes not applicable.

Planet mode category	Number of planets, N		
	4	5	6
Distinct planet modes	J+3	0	J+3
Degenerate planet modes	0	2(2J+3)	2(2J+3)
Type 2 planet modes	J+3	2(2J+3)	2(2J+3)
Type 3 planet modes	$\times$	$\times$	J+3
Type 4 planet modes	$\times$	$\times$	$\times$
Type 5 planet modes	$\times$	$\times$	$\times$
7	8	9	10
0	J+3	0	J+3
4(2J+3)	4(2J+3)	6(2J+3)	6(2J+3)
2(2J+3)	2(2J+3)	2(2J+3)	2(2J+3)
2(2J+3)	2(2J+3)	2(2J+3)	2(2J+3)
$\times$	J+3	2(2J+3)	2(2J+3)
$\times$	$\times$	$\times$	J+3

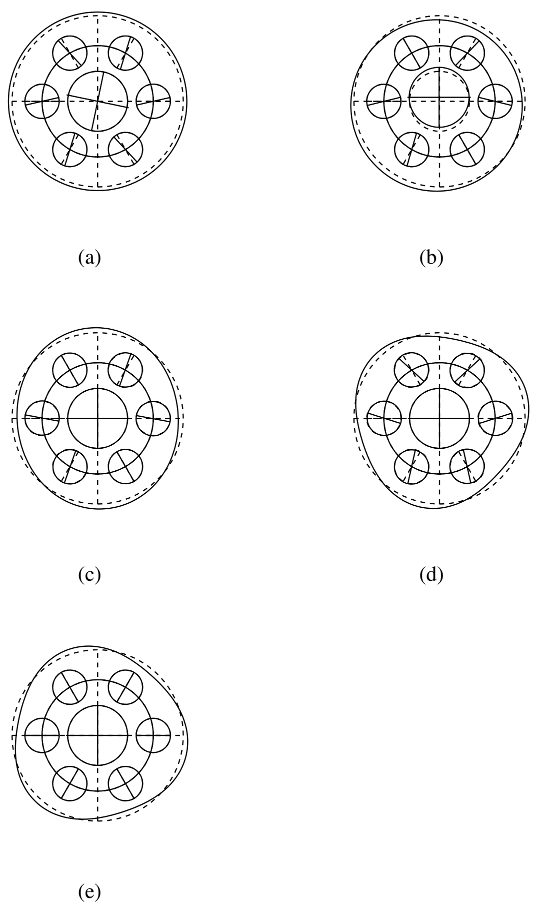

Figure 2. (a) Rotational mode (ω_1), (b) Translational mode ($\omega_{2,3}$), (c) Planet mode: degenerate ($\omega_{4,5}$), (d) Planet mode: distinct (ω_6), (e) Purely ring mode (ω_{12})Typical modes of a planetary gear. The system parameters are given in Table 1. Distinct planet modes as in (d) only exist for an even number of planets.